KTH Mathematics

Exam in SF2701 Financial Mathematics.
Wednesday August 16 2017 08.00-13.00.

Answers and brief solutions.

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1. (a) A slightly sloppy version of the first fundamental theorem is:
The market is free of arbitrage if and only if there exists a martingale measure.
It can be made precise for a concrete market.

(b) i. For the payoffs of the contracts we have that

$$\max\{K - P(T, S_T), 0\} = K - P(T, S_T) + \max\{P(T, S_T) - K, 0\}. \quad (1)$$

Now the price at time t of any T-claim X is given by

$$\Pi_t[X] = e^{-r(T-t)} E^Q \left[X | \mathcal{F}_t \right].$$

The price of the payoff on the left hand side of (1) is the price of a European put on a put, P_{put}, and the price of the payoff on the right hand side can be written

$$\begin{aligned} &\Pi_0[K - P(T, S_T) + \max\{P(T, S_T) - K, 0\}] = \\ &\Pi_0[K] - \Pi_0[P(T, S_T)] + \Pi_0[\max\{P(T, S_T) - K, 0\}] = \\ &e^{-rT} K - P(0, S_0) + C_{put} \end{aligned}$$

where C_{put} denotes the price of a European call on a put. To see that

$$\Pi_0[P(T, S_T)] = P(0, S_0)$$

use that

$$\begin{aligned} \Pi_0[P(T, S_T)] &= e^{-rT} E^Q \left[P(T, S_T) \right] \\ &= e^{-rT} E^Q \left[e^{-r(T_1 - T)} E^Q \left[\max\{K_1 - S_{T_1}, 0\} | \mathcal{F}_T \right] \right] \\ &= e^{-rT_1} E^Q \left[\max\{K_1 - S_{T_1}, 0\} \right] \\ &= P(0, S_0) \end{aligned}$$

where we have used the tower property that the smallest σ-algebra always wins.

The put-call parity is thus

$$P_{put} = e^{-rT} K - P(0, S_0) + C_{put}.$$

2. (a) We have that

$$\begin{aligned} T &= 8/12 = 2/3 \\ \Delta t &= T/2 = 1/3 \\ u &= e^{\sigma\sqrt{\Delta t}} \approx 1.1553 \\ d &= e^{-\sigma\sqrt{\Delta t}} \approx 0.8656 \end{aligned}$$

and the tree for the stock price is therefore

		62.7289
	54.2979	
50.0000		47.0000
	40.6830	
		35.2150

Now the option price tree can be computed using

$$q = \frac{e^{r\Delta t} - d}{u - d} \approx 0.5103,$$

and the discount factor

$$\frac{1}{e^{r\Delta t}} \approx \frac{1}{1.0134}$$

and the result is

		12.7289
	7.7637	
3.9095		0.0000
	0.0000	
		0.0000

In each node the value is obtained as

$$\max\{S_{t-} - 50, \frac{1}{1.0075}(q \cdot P^u + (1-q) \cdot P^d)\}$$

where S_{t-} is the stock price **just before dividend payment**, and P^u and P^d is the price of the option if the stock price goes up and down, respectively. Early exercise will be optimal in the node with option price 7.7637. The price of the option is thus 3.9095.

(b) The price of a call option will increase if there are no dividends, since the option is more likely to be in the money if the stock pays no dividends (the stock price will be higher).

3. (a) i. We have that the futures price (which is equal to the forward price, since interest rates are deterministic) is given by

$$\begin{aligned} F(0,T) &= \frac{\Pi_0(S_T)}{p(0,T)} = e^{rT} E^Q\left[\frac{S_T}{B_T}\right] = E^Q\left[S_T\right] \\ &= e^{(r-\delta)T} S_0 = e^{(0.02-0.04)*0.5} 100 = 99.0050 \end{aligned}$$

This futures price will make the initial value of the futures contract zero.

ii. Since we have computed the futures price we can use the Black -76 formula (which can be obtained from Black-Scholes formula using $s = e^{-r(T-t)} F_t$) with parameters

$$F_0 = 99.0050, \quad K = 95, \quad \sigma = 0.30, \quad r = 0.02, \quad T = 0.5.$$

The price of the call option is

$$c(0) = 10.2475.$$

Alternatively, one can use the Black-Scholes formula with parameters

$$S^{cy} = e^{-\delta T} S_0 = 98.0199, \quad K = 95, \quad \sigma = 0.30, \quad r = 0.02, \quad T = 0.5.$$

to obtain the same result.

(b) Let $P(t,s)$ denote the price of the portfolio at time t when $S_t = s$. We then have that

$$P(t,s) = c(t,s) + p(t,s).$$

Here $c(t,s)$ denotes the standard Black-Scholes price at time t of a European call option with exercise price K and expiry date T, when the current price of the underlying is s. The price of the corresponding put option is denoted by $p(t,s)$. By definition we have that

$$\Delta_P(t,s) = \frac{\partial P(t,s)}{\partial s}.$$

Using put-call-parity, that is

$$p(t,s) = Ke^{-r(T-t)} + c(t,s) - s,$$

we get

$$P(t,s) = c(t,s) + \left(Ke^{-r(T-t)} + c(t,s) - s\right) = 2c(t,s) + Ke^{-r(T-t)} - s.$$

The delta of the portfolio is thus

$$\Delta_P(t,s) = \frac{\partial P(t,s)}{\partial s} = 2\frac{\partial c(t,s)}{\partial s} - 1.$$

For a European call option in the standard Black-Scholes framework we have that

$$\Delta_{call} = \frac{\partial c(t,s)}{\partial s} = \Phi[d_1(t,s)].$$

Here Φ is the cumulative distribution function for the $N(0,1)$ distribution and

$$d_1(t,s) = \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}.$$

Therefore

$$\Delta_P(t,s) = 2\Phi[d_1(t,s)] - 1,$$

where $d_1(t,s)$ has been specified above.

4. (a) i. The zero coupon bond prices satisfy

$$p^K(0,T_i) = e^{-r(0,T_i)T_i}K.$$

Here we have $T_1 = 0.5$, $K = 100$, and $r(0,0.5) = 1.7\%$ and this yields the bond price

$$p^{100}(0,0.5) = 100e^{-0.017\cdot 0.5} = 99.1536.$$

ii. For $T_2 = 1$ $K = 100$, and $r(0,1) = 2\%$ we obtain the bond price

$$p^{100}(0,1) = 100e^{-0.02\cdot 1} = 98.0199.$$

iii. Fixed coupon bond prices are computed as

$$p_{fixed}(t) = \sum_{i=1}^{n} c_i p(t,T_i) + Kp(t,T_n).$$

For the two year coupon bond the coupon is $c^2 = 0.02 \cdot 0.5 \cdot 100 = 1$ and the formula reads

$$p_{fixed}(0) = 1p(0,0.5) + 1p(0,1) + 1p(0,1.5) + (1+100)p(0,2).$$

Using that $p(0,0.5) = 0.9991536$ (face value 1!), $p(0,1) = 0.980199$, $p(0,1.5) = e^{-0.023\cdot 1.5} = 0.966088$ and that $p(0,2) = e^{-0.025\cdot 2} = 0.951229$ this results in the coupon bond price

$$p_{fixed}(0) = 99.0120.$$

(b) The value of a forward rate agreement where you pay the rate $\bar{R}_s$ (quoted as a simple rate) over the interval $[S,T]$ on the principal K is given by

$$\Pi_0 = p(0,T)K\left[L(0;S,T) - \bar{R}_s\right](T-S).$$

For this exercise we have

$$\Pi_0 = p(0,2)10^6\left[L(0;1,2) - 0.03\right](2-1).$$

Here $L(0;1,2)$ denotes today's LIBOR rate for the second year. The LIBOR rate $L(0;S,T)$ is the same rate as the forward rate $f(0;S,T)$, only quoted as a simple rate. We thus have

$$1 + L(0;S,T)(T-S) = e^{f(0;S,T)(T-S)}.$$

We have the following relationship between forward rates and spot rates

$$r(t,T)(T-t) = r(t,S)(S-t) + f(t;S,T)(T-S).$$

Thus

$$f(t;S,T)(T-S) = r(t,T)(T-t) - r(t,S)(S-t)$$

and we get the one year forward rate for the second year is

$$f(0;1,2)(2-1) = r(0,2)\cdot 2.5 - r(0,1)\cdot 2 = 2.5\cdot 2 - 2.0\cdot 1 = 3.0\%,$$

In this exercise we need the LIBOR rate for the second year and this is obtained from

$$1 + L(0;1,2)\cdot 1 = e^{f(0;1,2)\cdot 1},$$

so

$$L(0;1,2) = e^{f(0;1,2)\cdot 1} - 1 = e^{0.03} - 1 \approx 0.0304545$$

The value of the forward contract is thus

$$\Pi_0 = p(0,2)10^6\left[L(0;1,2) - 0.03\right](2-1) = 0.951229\cdot 10^6(0.0304545 - 0.03)\cdot 1 \approx 432.366.$$

(c) One can compute the value of a swap as

$$\Pi_{swap} = p_{float} - p_{fixed}.$$

For the fixed coupon bond we have

$$p_{fixed} = \sum_{i=1}^{n} c_i p(t,T_i) + Kp(t,T_n).$$

For the floating rate bond we have that

$$p_{float} = K.$$

since the bond has just been issued. What is unknown to us in the formula for the price for the fixed coupon bond is $p(t,T)$. This can be found from the swap rates, since the swap rates are set so as to make the value of the fixed and the floating leg equal, i.e. for the one year swap

$$(c+K)p(0,1) = K.$$

Using that $c = R_s^1 \cdot \Delta T \cdot K$, we obtain

$$p(0,1) = \frac{1}{1 + R_s^1 \cdot \Delta T} = \frac{1}{1 + 0.017 \cdot 1} \approx 0.983284$$

The one year zero rate can now be found using that

$$p(0,T_i) = e^{-r(0,T_i)T_i}.$$

The one year zero rate is

$$r(0,1) = -\ln p(0,1) = 0.01685,$$

or 1.685%.
Similarly for the two year swap we have that

$$cp(0,1) + (c+K)p(0,2) = K.$$

Using that $c = R_s^2 \cdot \Delta T \cdot K$, we therefore obtain

$$p(0,2) = \frac{1 - R_s^2 \Delta T p(0,1)}{1 + R_s^2 \Delta T} = \frac{1 - 0.02 \cdot 0.983284}{1 + 0.02} \approx 0.961112$$

and the two year zero rate is

$$r(0,2) = -\frac{1}{2} \ln p(0,2) = 0.1983,$$

or 1.983%.

5. (a) If we can show that

$$X_{ave\_price\_call} + X_{ave\_strike\_call} - X_{call} = X_{ave\_price\_put} + X_{ave\_strike\_put} - X_{put} \quad (2)$$

where

$$X_{call} = \max\{S_T - K, 0\}, \text{ and } \quad X_{put} = \max\{K - S_T, 0\}$$

the desired result will follow, since the price at time t of any T-claim X is given by

$$\Pi_t[X] = e^{-r(T-t)} E^Q\left[X|\mathcal{F}_t\right]$$

and conditional expectation is a linear operator.
The payoff of the call side is

$$X_{ave\_strike\_call} + X_{ave\_price\_call} - X_{call} =$$

$$\max\{S_T - S_T^{ave}, 0\} + \max\{S_T^{ave} - K, 0\} - \max\{S_T - K, 0\},$$

and the payoff of the put side is

$$X_{ave\_strike\_put} + X_{ave\_price\_put} - X_{put} =$$

$$\max\{S_T^{ave} - S_T, 0\} + \max\{K - S_T^{ave}, 0\} - \max\{K - S_T, 0\}.$$

For the payoffs we have the following table:

	Payoff call side	Payoff put side
$K > S_T > S_T^{ave}$	$S_T - S_T^{ave} + 0 - 0$	$0 + K - S_T^{ave} - (K - S_T)$
$K > S_T^{ave} > S_T$	$0 + 0 + 0$	$S_T^{ave} - S_T + K - S_T^{ave} - (K - S_T)$
$S_T > S_T^{ave} > K$	$S_T - S_T^{ave} + S_T^{ave} - K - (S_T - K)$	$0 + 0 + 0$
$S_T > K > S_T^{ave}$	$S_T - S_T^{ave} + 0 - (S_T - K)$	$0 + K - S_T^{ave} - 0$
$S_T^{ave} > S_T > K$	$0 + S_T^{ave} - K - (S_T - K)$	$S_T^{ave} - S_T + 0 - 0$
$S_T^{ave} > K > S_T$	$0 + S_T^{ave} - K - 0$	$S_T^{ave} - S_T + 0 - (K - S_T)$

Therefore the payoff of the call side is always equal to the payoff of the put side, and the result follows.

(b) For the binary cash-or-nothing put we have

$$\begin{aligned}
\Pi_t[\mathrm{BCP}_T] &= e^{-r(T-t)} E^Q \left[K_1 I_{\{S_T < K_2\}} \middle| \mathcal{F}_t \right] \\
&= e^{-r(T-t)} K_1 Q_{t,s} \left(S(T) < K_2 \right) \\
&= e^{-r(T-t)} K_1 Q \left(s e^Z \leq K_2 \right),
\end{aligned}$$

where $Z \in \mathrm{N}\left((r - \sigma^2/2)(T-t), \sigma^2(T-t)\right)$. Rewriting a bit gives

$$\begin{aligned}
\Pi_t[\mathrm{BCC}_T] &= e^{-r(T-t)} K_1 N \left(\frac{1}{\sigma\sqrt{T-t}} \left\{ \ln \left\{ \frac{K_2}{s} \right\} - \left(r - \frac{1}{2}\sigma^2 \right)(T-t) \right\} \right) \\
&= e^{-r(T-t)} K_1 N \left(\frac{1}{\sigma\sqrt{T-t}} \left\{ -\ln \left\{ \frac{s}{K_2} \right\} - \left(r - \frac{1}{2}\sigma^2 \right)(T-t) \right\} \right).
\end{aligned}$$

If you like this can be written as $K_1 e^{-r(T-t)} N(-d_2)$, where you should use K_2 in the term d_2.