Exam in SF2701 Financial Mathematics.
Wednesday May 302018 14.00-19.00.

Answers and brief solutions.

1. (a) Replicating portfolio. The number of stocks in the replicating portfolio is $y=$ $\Delta$ i.e.

$$
y=\Delta=\frac{\Delta \Pi}{\Delta S}=\frac{0-4}{54-47}=\frac{-4}{7} .
$$

To find the amount of cash $x$ you should have in the bank account solve

$$
x e^{r \Delta t}+y S_{t+\Delta t}^{u}=\Pi^{u}(t+\Delta t)
$$

with numbers

$$
x e^{0.05 \cdot 0.25}+\frac{-4}{7} \cdot 54=0 .
$$

This yields $x=e^{-0.05 \cdot 0.25} 216 / 7$ and the replicating portfolio is therefore $(x, y)=$ (30.47, -4/7).
(This means that the value of the option at time $t=0$, which is equal to the value of the replicating portfolio at time $t=0$, is

$$
\left.x+y S_{0}=e^{-0.05 \cdot 0.25} 216 / 7-\frac{4}{7} \cdot 50 \approx 1.90 .\right)
$$

(b) i. The payoff from the call option and the put option look as follows


Since the call has been bought and the put has been sold to create the synthetic long stock, the payoff from the synthetic long stock is given by


In the figures above $K=100$ has been used, but hopefully it should be clear what the payoff looks like for a general $K$.
ii. If we denote by $C\left(t, S_{t}\right)$ the price at time $t$ of a European call option with strike price $K$ and expiry date $T$ written on the stock with price $S_{t}$ at time $t$, and by $P\left(t, S_{t}\right)$ the price at time $t$ of a European put option with the same strike price and expiry date as the call, and also having the stock as underlying, then according to put-call parity we have

$$
P\left(t, S_{t}\right)=K e^{-r(T-t)}+C\left(t, S_{t}\right)-S_{t}
$$

The price the synthetic long stock is therefore

$$
C\left(t, S_{t}\right)-P\left(t, S_{t}\right)=K e^{-r(T-t)}-S_{t}
$$

or with numbers

$$
S_{t}-K e^{-r(T-t)}=86-85 e^{-0.02 \cdot 3 / 12}=1.4239
$$

2. (a) We have that

$$
\begin{aligned}
T & =6 / 12=1 / 2 \\
\Delta t & =T / 2=1 / 4 \\
u & =e^{\sigma \sqrt{\Delta t}} \approx 1.0513 \\
d & =e^{-\sigma \sqrt{\Delta t}} \approx 0.9512
\end{aligned}
$$

and the tree for the stock price is therefore
121.5688
115.6398
$110.0000 \quad 110.0000$
104.6352
99.5321

The continuous dividend yield $\delta$, should be equal to the US interest rate, i.e.
$\delta=0.04$, whereas the rate $r$ should be set to the Japanese interest rate, i.e. $r=0.01$. Now the option price tree can be computed using

$$
q=\frac{e^{(r-\delta) \Delta t}-d}{u-d} \approx 0.4128
$$

Note that the dividend yield comes in to play when computing the martingale probabilities. For pricing we also need the discount factor

$$
\frac{1}{e^{r \Delta t}} \approx \frac{1}{1.0025}
$$

and the result is

$$
16.5688
$$

10.6398

$$
\begin{array}{lll}
5.5873 & & 5.0000 \\
& 2.0589 & \\
& & 0.0000
\end{array}
$$

In each node, that is not an end node, the value is obtained as

$$
\max \left\{S_{t}-105, \frac{1}{1.0025}\left(q \cdot P^{u}+(1-q) \cdot P^{d}\right)\right\}
$$

where $S_{t}$ is the current stock price, and $P^{u}$ and $P^{d}$ is the price of the option if the stock price goes up and down, respectively. Early exercise will be optimal in the node with option price 10.6398. The price of the option is thus 5.5873.
(b) If we denote the option price by $\Pi$ we have that

$$
\Delta=\frac{\partial \Pi}{\partial s} \approx \frac{\Delta \Pi}{\Delta s} .
$$

This gives us

$$
\Delta=\frac{10.6398-2.0589}{115.6398-104.6352} \approx 0.7798
$$

3. (a) i. We have that the forward price is given by

$$
f(0, T)=\frac{\Pi_{0}\left(S_{T}\right)}{p(0, T)}=e^{r T} E^{Q}\left[\frac{S_{T}}{B_{T}}\right] .
$$

This forward price will make the initial value of the forward contract zero.
To compute the expectation note that we have the formula

$$
S_{0}=E^{Q}\left[\frac{S_{T}}{B_{T}}+\sum_{t_{i} \leq T} \frac{\Delta D_{t_{i}}}{B_{t_{i}}}\right]
$$

Here we can insert any $T$ and therefore we have that

$$
\begin{equation*}
S_{0}=E^{Q}\left[\frac{S_{9 / 12}}{B_{9 / 12}}+\frac{\Delta D_{3 / 12}}{B_{3 / 12}}\right] \tag{1}
\end{equation*}
$$

but also

$$
\begin{equation*}
S_{0}=E^{Q}\left[\frac{S_{3 / 12-}}{B_{3 / 12-}}\right] \tag{2}
\end{equation*}
$$

If we use that $\Delta D_{3 / 12}=0.05 S_{3 / 12-}$ and insert this and (2) into (1), we obtain

$$
S_{0}=E^{Q}\left[\frac{S_{9 / 12}}{B_{9 / 12}}+\frac{0.05 S_{3 / 12-}}{B_{3 / 12-}}\right]=E^{Q}\left[\frac{S_{9 / 12}}{B_{9 / 12}}\right]+0.05 S_{0}
$$

or

$$
E^{Q}\left[\frac{S_{9 / 12}}{B_{9 / 12}}\right]=(1-0.05) S_{0}
$$

The forward price is therefore

$$
f(0,9 / 12)=e^{0.03 \cdot 9 / 12}(1-0.05) 100 \approx 97.1617
$$

ii. Since we have just computed the forward price and interest rates are deterministic, which means that forward prices and futures prices are equal, we can use the Black - 76 formula (which can be obtained from Black-Scholes formula using $s=e^{-r(T-t)} F_{t}$ ) with parameters

$$
F_{0}=97.1617, \quad K=95, \quad \sigma=0.30, \quad r=0.03, \quad T=0.75
$$

or the Black-Scholes formula with parameters

$$
s=(1-\delta) S_{0}=95, \quad K=95, \quad \sigma=0.30, \quad r=0.03, \quad T=0.75
$$

The price of the call option is therefore

$$
c(0)=10.8027
$$

To obtain the put price we use put-call parity for futures options

$$
p_{f u t}(0)=e^{-r T}\left(K-F_{0}\right)+c_{f u t}(0)
$$

(or put-call parity for spot options, but with $s=(1-\delta) S_{0}$ ). This yields that the price of the put option is

$$
p(0)=8.6891
$$

Here we have used that since the maturity of the spot option is equal to the delivery time of the forward contract we have $f(0,075,0.75)=$ $F(0.75,0.75)=S(0.75)$, so the payoff from the futures option is the same as the payoff from the spot option.
(b) If we denote the price of a derivative written on the underlying stock by $\Pi$ we have by definition that

$$
\Delta=\frac{\partial \Pi}{\partial s}
$$

In the case of the cash-or-nothing call we therefore get

$$
\begin{aligned}
\Delta_{c n} & =\frac{\partial \Pi_{c n}}{\partial s} \\
& =\frac{\partial}{\partial s} e^{-r(T-t)} K \Phi\left[d_{2}(t, s)\right] \\
& =e^{-r(T-t)} K \Phi^{\prime}\left[d_{2}(t, s)\right] \frac{1}{\sigma \sqrt{T-t}} \frac{1}{s} \\
& =\frac{e^{-r(T-t)} K \varphi\left(d_{2}(t, s)\right)}{\sigma s \sqrt{T-t}}
\end{aligned}
$$

where $\varphi(x)$ denotes the density function of a standard normal distribution. So

$$
\Delta_{c n}=\frac{e^{-r(T-t)} K \varphi\left(d_{2}\left(t, S_{t}\right)\right)}{\sigma S_{t} \sqrt{T-t}}
$$

where

$$
d_{2}(t, s)=\frac{1}{\sigma \sqrt{T-t}}\left\{\ln \left(\frac{s}{K}\right)+\left(r-\frac{1}{2} \sigma^{2}\right)(T-t)\right\}
$$

4. (a) The zero coupon bond prices satisfy

$$
p^{K}\left(0, T_{i}\right)=e^{-r\left(0, T_{i}\right) T_{i}} K
$$

Here we have $T_{1}=0.5, K=100$, and $p^{100}(0,0.5)=99.0050$ and this yields the zero rate

$$
r(0,0.5)=2.0000
$$

Fixed coupon bond prices are computed as

$$
p_{\text {fixed }}(t)=\sum_{i=1}^{n} c_{i} p\left(t, T_{i}\right)+K p\left(t, T_{n}\right)
$$

For the one year coupon bond the coupon is $c^{1}=0.03 \cdot 0.5 \cdot 100=1.5$ and the formula reads

$$
100.4790=1.5 p(0,0.5)+(1.5+100) p(0,1)
$$

Using that $p(0,0.5)=0.99005$ (face value 1 !) we obtain that $p(0,1)=0.9753$ and this results in the zero rate

$$
r(0,1)=2.5000 \%
$$

Finally, for the bond with maturity in 18 months we find that the coupon is $c^{1.5}=0.03 \cdot 1 \cdot 100=3$ and the formula reads

$$
101.4379=3 p(0,0.5)+(3+100) p(0,1.5)
$$

Using that $p(0,0.5)=0.99005$ we obtain that $p(0,1.5)=0.9560$ this results in the zero rate

$$
r(0,1.5)=3.0000 \%
$$

(b) We have the following relationship between forward rates and spot rates

$$
r(t, T)(T-t)=r(t, S)(S-t)+f(t ; S, T)(T-S)
$$

Thus

$$
f(t ; S, T)(T-S)=r(t, T)(T-t)-r(t, S)(S-t)
$$

and we get the one year forward rate starting in six months is

$$
f(0 ; 0.5,1.5)(1.5-0.5)=r(0,1.5) \cdot 1.5-r(0,0.5) \cdot 0.5=3.0 \cdot 1.5-2.0 \cdot 0.5=3.5 \%
$$

The simple rate is obtained from

$$
1+L(0 ; 0.5,1.5) \cdot 1=e^{f(0 ; 0.5,1.5) \cdot 1}
$$

So

$$
L(0 ; 0.5,1.5)=e^{f(0 ; 0.5,1.5) \cdot 1}-1=e^{0.035}-1 \approx 0.0356197=3.56197 \%
$$

The general relationship between forward rates is if $S<U<T$

$$
f(0 ; S, T)(T-S)=f(0 ; S, U)(U-S)+f(0 ; U, T)(T-U)
$$

One way of proving it is to use the definition of the forward rate

$$
f(0 ; S, T)=-\frac{\ln p(0, T)-\ln p(0, S)}{T-S}
$$

We then have that

$$
\begin{aligned}
& f(0 ; S, U)(U-S)+f(0 ; U, T)(T-U)= \\
& =-\frac{\ln p(0, U)-\ln p(0, S)}{U-S}(U-S)-\frac{\ln p(0, T)-\ln p(0, U)}{T-U}(T-U)= \\
& =-\frac{\ln p(0, T)-\ln p(0, S)}{T-S}(T-S)= \\
& =f(0 ; S, T)(T-S)
\end{aligned}
$$

(c) For the forward price we have that

$$
f\left(0 ; 1, p_{\text {fixed }}(1)\right)=\frac{\Pi_{0}\left(p_{\text {fixed }}(1)\right)}{p(0,1)}
$$

The denominator is $p(0,1)=0.9753$ computed in (a) and the numerator is

$$
\Pi_{0}\left(p_{\text {fixed }}(1)\right)=(3+100) p(0,1.5)=103 * 0.955997573=98.46775
$$

where $p(0,1.5)=0.9560$ was computed in (a). The forward price is therefore

$$
f\left(0 ; 1, p_{\text {fixed }}(1)\right)=\frac{\Pi_{0}\left(p_{\text {fixed }}(1)\right)}{p(0,1)}=\frac{98.46775}{0.9753}=100.96
$$

5. (a) i. The payoff functions for the gap call and put are given by



In the above figure $K_{s}=100$ and $K_{t}=120$ have been used.
ii. For the payoff functions it holds that

$$
\left(S_{T}-K_{s}\right) I\left\{S_{T}>K_{t}\right\}+\left(K_{s}-S_{T}\right) I\left\{S_{T}<K_{t}\right\}=S_{T}-K_{s}
$$

where the notation $I\{s>K\}$ is used for the indicator function

$$
I\{s>K\}= \begin{cases}1, & \text { if } s>K \\ 0, & \text { otherwise }\end{cases}
$$

The relation between the payoffs will carry over to the prices and it then follows that we have the put-call parity

$$
c_{g a p}(t)-p_{g a p}(t)=S_{t}-e^{-r(T-t)} K_{s}
$$

Note that the parity does not depend on the trigger price $K_{t}$.
(b) Developing the square we have

$$
X=\phi\left(S_{T}\right)=\left(S_{T}^{2}-2 K S_{T}+K^{2}\right)^{2} I\left\{S_{T}>K\right\}
$$

The two last terms you will recognize from the payoff of a standard call option (multiplied by $-2 K$ and $-K$, respectively).
The price of the first term is given by

$$
\Pi_{t}=e^{-r(T-t)} E^{Q}\left[S_{T}^{2} I_{\left\{S_{T}>K\right\}} \mid \mathcal{F}_{t}\right]
$$

Since $S_{T}=S_{t} e^{Z}$ where $Z \in N\left(\left(r-\sigma^{2} / 2\right)(T-t), \sigma^{2}(T-t)\right)$ this can be written as

$$
\Pi_{t}=e^{-r(T-t)} \int_{\ln \left\{\frac{K}{S_{t}}\right\}}^{\infty} S_{t}^{2} e^{2 z} \varphi(z) d z
$$

where $\varphi$ denotes the density of a $N\left(\left(r-\sigma^{2} / 2\right)(T-t), \sigma^{2}(T-t)\right)$-distribution. Now use that the density function for a $N\left(m, \sigma^{2}\right)$-distributed random variable is $\varphi(z)=e^{-(z-m)^{2} /\left(2 \sigma^{2}\right)} /(\sigma \sqrt{2 \pi})$, and then complete the square in the exponent. This yields

$$
\Pi_{t}=e^{-r(T-t)} e^{\left(2 r+\sigma^{2}\right)(T-t)} S_{t}^{2} \int_{\ln \left\{\frac{K}{S_{t}}\right\}}^{\infty} \psi(u) d u
$$

where $\psi$ denotes the density of a $N\left(\left(r+3 \sigma^{2} / 2\right)(T-t), \sigma^{2}(T-t)\right)$-distribution. If we let $U$ denote a $N\left(\left(r+3 \sigma^{2} / 2\right)(T-t), \sigma^{2}(T-t)\right)$-distributed random variable. Then we have that

$$
\begin{aligned}
\Pi_{t} & =e^{\left(r+\sigma^{2}\right)(T-t)} S_{t}^{2} Q\left(U>\ln \left\{\frac{K}{S_{t}}\right\}\right) \\
& =e^{\left(r+\sigma^{2}\right)(T-t)} S_{t}^{2}\left[1-\Phi\left(\frac{1}{\sigma \sqrt{T-t}}\left\{\ln \left\{\frac{K}{S_{t}}\right\}-\left(r+\frac{3}{2} \sigma^{2}\right)(T-t)\right\}\right)\right] \\
& =e^{\left(r+\sigma^{2}\right)(T-t)} S_{t}^{2} \Phi\left(\frac{1}{\sigma \sqrt{T-t}}\left\{\ln \left\{\frac{S_{t}}{K}\right\}+\left(r+\frac{3}{2} \sigma^{2}\right)(T-t)\right\}\right),
\end{aligned}
$$

where we have used one of the hints to obtain the last equality.
All in all the price is thus given by

$$
\Pi^{p c}=S_{t}^{2} e^{\left(r+\sigma^{2}\right)(T-t)} \Phi\left(d_{0}\left(t, S_{t}\right)\right)-2 K s_{t} \Phi\left(d_{1}\left(t, S_{t}\right)\right)+e^{-r(T-t)} K^{2} \Phi\left(d_{2}\left(t, S_{t}\right)\right)
$$

where

$$
d_{i}(t, s)=\frac{\ln (s / K)+\left[r+(3 / 2-i) \sigma^{2}\right](T-t)}{\sigma \sqrt{T-t}} \quad \text { for } i=0,1,2
$$

