KTH Mathematics

Examination in SF2701 Financial Mathematics, August 27, 2008, 14:00–19:00.

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Allowed technical aids: calculator.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

Interest rates are continuously compounded. Figure 1 shows a plot of the standard normal distribution function.

GOOD LUCK!

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Problem 1

Maturity (years)	0.5	1	1.5	2
Annual coupon ($)	0	0	8	12
Bond price ($)	94.9	90.0	96.0	101.6
Face value ($)	100	100	100	100

Table 1: Specification of four different bonds.

Consider the bonds specified in Table 1. Half of the annual coupon is paid every six months, until and including the time of maturity, and the first coupon payment is in six months.
Determine the current spot price of a contract which in 18 months pays the holder the floating interest rate that prevails between the time points 12 months and 18 months on the amount $100. The floating rate is the zero-coupon rate determined in 12 months from now for cash received in 18 months from now. (10 p)

Problem 2

Determine the current spot price of a European put option on a share whose current spot price is $100 with maturity in one year with strike price $104. The share's volatility is 20% in one year, the interest rate is 5% a year. The share does not pay dividends. Use a binomial tree with time interval three months (binomial trees should be constructed as in the lecture notes). (10 p)

Problem 3

Determine the current spot price of a European put option on a share whose current spot price is $100 with maturity in one year with strike price $104. The share's volatility is 20% in one year, the interest rate is 5% a year.

(a) The share pays no dividend. (4 p)

(b) The share pays a dividend of $4 in three months. (6 p)

Use Black's formula for put options: $p = Z_t(K\Phi(-d_2) - G\Phi(-d_1))$, where $d_1 = \ln(G/K)/(\sigma\sqrt{t}) + \sigma\sqrt{t}/2$ and $d_2 = d_1 - \sigma\sqrt{t}$.

Problem 4

Let X be the spot price in \$ of 15′000 pounds of orange juice on April 27, 2009 (time t). We assume that under the forward probability distribution,

$$X = Ae^{\sigma\sqrt{t}z}, \quad \text{where } z \text{ is } N(0,1)\text{-distributed.}$$

We assume that the volatility of X is 30% in one year, the interest rate is 5% a year, and the current futures price for delivery of 15′000 pounds of orange juice on April 27, 2009, is \$10′000.
What is the current spot price of a derivative contract which pays \$100 if the spot price of 15′000 pounds of orange juice on April 27, 2009, is above \$11′000 and nothing otherwise? (10 p)

Problem 5

McDanolds has negotiated a contract to buy 150′000 pounds of orange juice in April 2009 at the spot price at that time. To hedge the uncertain cost McDanolds wants to take a position in orange juice futures contracts so that the variance of the effective price at which McDanolds buys (the hedged position) is minimized. Each futures contract is for delivery of 15′000 pounds of orange juice in May 2009. The current spot- and futures prices are 90 and 70 cents per pound, respectively.
The correlation coefficient between the spot- and futures price percentage returns (the April 2009 spot(futures) price divided by the current spot(futures) price) is assumed to be 0.7 and the standard deviations of the spot- and futures price percentage returns are assumed to be 0.4 and 0.6, respectively.
What futures position (long/short and size) should the McDanolds take and what is the expected effective price per pound at which McDanolds buys? (10 p)

Hint: The regression coefficient of Y onto X is $\mathrm{Cov}(X,Y)/\mathrm{Var}(X)$.

Problem 6

Consider a market with a constant interest rate of 5% a year and a share which pays no dividend. Half a year ago the spot price of the share was \$50 and at that time you entered a forward contract for delivery of one share to you in one year (from that time). Now the spot price of the share has changed to \$60. What is the current market value of your forward contract? (10 p)

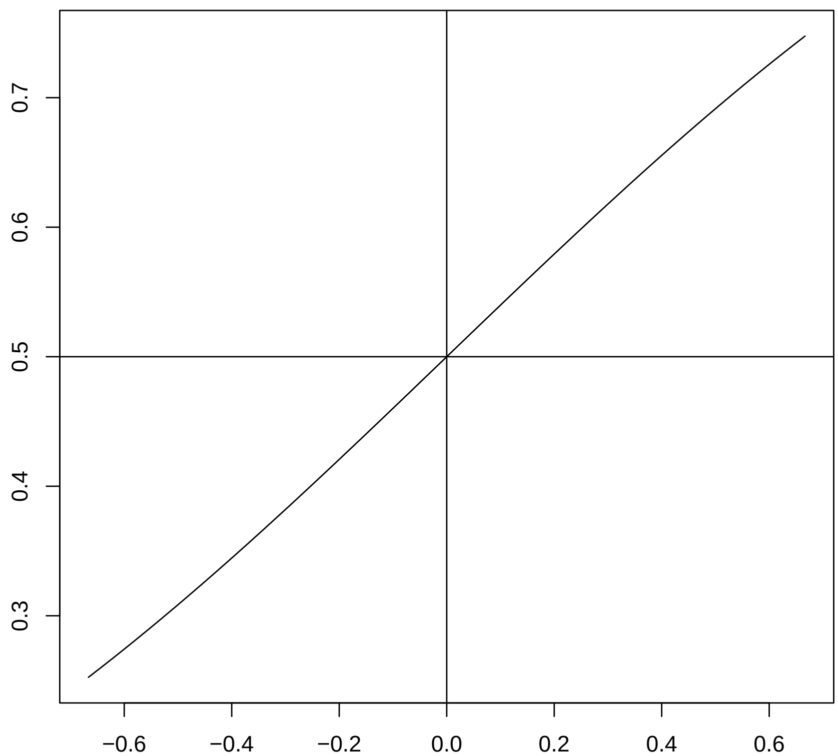

Figure 1: Plot of the standard normal distribution function Φ.

Problem 1

Write r_t for the t-year zero rate. We have

$$
\begin{aligned}
r_{0.5} &= -2\ln\left(\frac{94.9}{100}\right) = 0.1046930 \\
r_1 &= -\ln\left(\frac{90}{100}\right) = 0.1053605 \\
r_{1.5} &= -\frac{2}{3}\ln\left(\frac{96 - 4e^{-r_{0.5}0.5} - 4e^{-r_1}}{104}\right) = 0.1068093 \\
r_2 &= -\frac{1}{2}\ln\left(\frac{101.6 - 6e^{-r_{0.5}0.5} - 6e^{-r_1} - 6e^{-r_{1.5}1.5}}{106}\right) = 0.1080802.
\end{aligned}
$$

The present value of the floating interest rate on \$100 is (see the section of interest rate swaps in the lecture notes) $\$100(Z_1 - Z_{1.5}) \approx \$4.804$.

Problem 2

0	1	2	3	4
G_0	G_0u	G_0u^2	G_0u^3	G_0u^4
	G_0d	G_0du	G_0du^2	G_0du^3
		G_0d^2	G_0d^2u	$G_0d^2u^2$
			G_0d^3	G_0d^3u
				G_0d^4

0	1	2	3	4
S_0	$S_0u\gamma$	$S_0u^2\gamma^2$	$S_0u^3\gamma^3$	$S_0u^4\gamma^4$
	$S_0d\gamma$	$S_0du\gamma^2$	$S_0du^2\gamma^3$	$S_0du^3\gamma^4$
		$S_0d^2\gamma^2$	$S_0d^2u\gamma^3$	$S_0d^2u^2\gamma^4$
			$S_0d^3\gamma^3$	$S_0d^3u\gamma^4$
				$S_0d^4\gamma^4$

0	1	2	3	4
100	111.3500	123.9883	138.0609	153.7309
	91.16568	101.513	113.0347	125.8642
		83.11181	92.54501	103.0489
			75.76945	84.36929
				69.07573

0	1	2	3	4
7.083905	2.707059	0.2319043	0	0
	11.63896	5.250315	0.4696426	0
		18.32041	10.16307	0.9511
			26.93864	19.63071
				34.92427

The first tree is the tree of forward prices, where G_0 is the current forward price for delivery of the share in one year and $u = 1 + \tanh(0.1) = 1.099668$ and $d = 1 - \tanh(0.1) = 0.900332$. The second tree is the corresponding tree of spot prices, where $\gamma = \exp\{r\Delta t\} = 1.012578$. The third tree is the same tree with numerical values inserted. The fourth tree is the tree of put option prices. Hence, the put option price is \$7.083905.

Problem 3

(a) $G_0^{(1)}[S_1] = 105.1271$, $K = 104$, $t = 1$, $\sigma = 0.2$, $d_1 = 0.1538964$, $d_2 = -0.04610357$. The option price is \$7.398263.

(b) As in (a) but with $G_0^{(1)}[S_1] = 100.9743$, $d_1 = -0.04762625$ and $d_2 = -0.2476262$. The option price is \$9.288797.

Problem 4

We have $t = 8/12$, $\sigma = 0.3$ and $G := G_0^{(t)}[X] = 10'000$. Moreover, $X = G\exp\{-\frac{1}{2}\sigma^2 t + \sigma\sqrt{t}z\}$. Hence, with $f(x) = I_{(11'000,\infty)}(x)$,

$$\begin{aligned}
P_0^{(t)}[100f(X)] &= e^{-0.05t}100\,\mathrm{E}^{(t)}[f(X)] \\
&= e^{-0.05t}100\,\mathrm{P}^{(t)}[\exp\{-\sigma^2 t/2 + \sigma\sqrt{t}z\} > 11/10] \\
&= e^{-0.05t}100\,\mathrm{P}^{(t)}[z > (\ln(11/10) + \sigma^2 t/2)/(\sigma\sqrt{t})] \\
&= e^{-0.05t}100(1 - \Phi(0.496752863)) \\
&= 29.82837
\end{aligned}$$

Problem 5

Let today be time 0, let time t be the contracted date in April. The hedge should be a long position in the orange juice futures contracts, closed out at time t. We have to determine the size of the position. In order to do this we may determine the regression coefficient β of the percentage return of the orange juice spot price onto the percentage return of the orange juice futures price:

$$S_t/S_0 = \beta F_t/F_0 + e. \tag{1}$$

From standard theory of linear regression we know that

$$\beta = \frac{\mathrm{Cov}(S_t/S_0, F_t/F_0)}{\mathrm{Var}(F_t/F_0)} = \mathrm{Corr}(S_t/S_0, F_t/F_0)\sqrt{\frac{\mathrm{Var}(S_t/S_0)}{\mathrm{Var}(F_t/F_0)}} = 0.7\frac{0.4}{0.6} = \frac{7}{15}.$$

This choice of β minimizes the variance $\mathrm{Var}(e)$ of the residual term e. Moreover, it is easily checked that this choice of β implies that $\mathrm{Cov}(F_t/F_0, e) = 0$. The effective price per pound at time t for the orange juice is the spot price at time t minus the gain (or loss) per pound from the long futures position: $S_t - kn(F_t - F_0)$, where $n = 15'000$ and k is the number of futures contracts per pound of orange juice bought. Using (1) we can write this price as

$$\begin{aligned}
S_t + kn(F_0 - F_t) &= \beta\frac{S_0}{F_0}F_t + S_0 e + kn(F_0 - F_t) \\
&= \left(\beta\frac{S_0}{F_0} - kn\right)F_t + knF_0 + S_0 e \\
&= \beta S_0 + S_0 e
\end{aligned}$$

when choosing

$$k = \frac{\beta S_0}{nF_0} = \frac{7}{15}\frac{9}{7}\frac{1}{n} = \frac{3}{5n}.$$

Since McDanolds will buy 150′000 pounds of orange juice, McDanolds should take a long position of $150'000k = 6$ futures contracts. Since e has zero expected value, then expected effective price is $\beta S_0 = 42$ cents per pound.

Problem 6

The forward price contracted six month ago was (time measured in years from that time)

$$G := G_0^{(1)}[S_1] = e^{0.05} P_0^{(1)}[S_1] = \$e^{0.05} 50 = \$52.56355482.$$

Today, the cash flow of the forward contract is (time measured in years from today) $S_{0.5} - G$ delivered in six months. The present value of this cash flow is

$$P_0^{(0.5)}[S_{0.5} - G] = S_0 - e^{-0.025} G = \$(60 - 51.26575603) = \$8.734243974.$$