KTH Mathematics

Examination in SF2701 Financial Mathematics, August 24, 2010, 14:00–19:00.

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Allowed technical aids: calculator.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

Interest rates are continuously compounded. Binomial trees should be constructed as in the lecture notes.

Good luck!

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Problem 1

Consider an economy in discrete time with a time step of three months. In half a year from now you will need 1 kg of the newly discovered metal Platinuf. You can take a long position in a futures contract for delivery in six months of 1 kg of Platinuf. The current futures price is 1000 in the currency of this economy. The price of a zero-coupon bond maturing in three months is known today. The price in three months from now of a zero-coupon bond maturing in six months from now is not known today but the correlation coefficient, with respect to the futures distribution, between this future bond price and the spot price of Platinuf six months from now is 0.15. You are contacted by a serious Platinuf dealer who offers you a forward contract for delivery of 1 kg of Platinuf to you in half a year. The forward price would be 1000. Determine whether the futures contract is a better deal than the forward contract. (10 p)

Problem 2

The current one-year interest rate is 3%, i.e. a zero-coupon bond maturing in one year with face value 1 costs $e^{-0.03}$ today. Let r be the (random) one-year interest rate one year from now, i.e. e^{-r} is the (random) price one year from now of a zero-coupon bond maturing in two years from now. With respect to the two-year forward probability distribution, $r = 0.03+\sigma Z$ with Z is standard normally distributed. The current two-year interest rate is 4% a year. Determine σ. (10 p)

Problem 3

Determine the price of an American put option with maturity in nine months and with strike price 104 on a share whose current spot price is 100. The share's volatility is 20% in one year, the interest rate is 5% a year. The share does not pay dividends. Use a binomial tree with a time step of three months.

You may use the following information: $\tanh(0.05) = 0.04995837$, $\tanh(0.1) = 0.099668$, $\tanh(0.2) = 0.1973753$. (10 p)

Problem 4

Determine the price of a European derivative on the spot price one year from now of one share of a dividend paying stock. The payoff function $f(x)$ of the derivative is $f(x) = x^3$. The current share price is 100 and the interest rate is 5% a year. The share pays a dividend of 5 in four months. Use Black's model and assume that the share's volatility is 20% in one year. (10 p)

Problem 5

A company will buy orange juice in April 2011 at the spot price at that time. To hedge the uncertain cost the company wants to take a position in orange juice futures contracts so that the variance of the effective price (including the gain/loss from the futures position) for buying the orange juice is minimized. Each futures contract is for delivery of $15'000$ pounds of orange juice in April 2011. The current spot- and futures prices are 90 and 70 cents per pound, respectively. The expected value and standard deviation of the April 2011 spot price S are 80 and 20 cents, respectively. The number of pounds X of orange juice it needs to buy is unknown today but can be modeled as the random variable with expected value 150'000.
Determine an expression for the futures position that the company should take. The expression should be as simple as possible but may include the random variables X and S. You may assume that the maturity date of the futures contract coincides with the date on which the company will buy the orange juice. (5 p)

Now assume that the quantity that that company needs to buy in April 2011 is independent of the spot price at that time. What futures position (long/short and size) should the company take? (5 p)

Problem 6

Determine the forward price in Euro on a Lithuanian share that today costs 100 Litas. The maturity of the forward contract is in one year and in 8 months from now the share pays a dividend of 5% of its spot price at that time. The Euro rate is currently 3% per year and the Lithuanian rate is 5% per year. One Euro costs 3.5 Litas today. (10 p)

Problem 1

Note that

$$\mathrm{Cor}(X,Y) > 0 \text{ means } \frac{\mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y]}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} > 0 \text{ which implies } \mathrm{E}[XY] > \mathrm{E}[X]\,\mathrm{E}[Y].$$

Let X be the spot price of 1 kg of Platinuf six months from now. With $Y = e^{-r_2}$ and $\mathrm{E} = \widehat{\mathrm{E}}$ we get

$$\begin{aligned} G_0^{(2)}[X] &= Z_2^{-1} P_0^{(2)}[X] \\ &= Z_2^{-1} F_0^{(2)}[Xe^{-r_1-r_2}] \\ &= Z_2^{-1} \widehat{\mathrm{E}}[Xe^{-r_1-r_2}] \\ &> Z_2^{-1} \widehat{\mathrm{E}}[X] \widehat{\mathrm{E}}[e^{-r_1-r_2}] \\ &> Z_2^{-1} F_0^{(2)}[X] F_0^{(2)}[e^{-r_1-r_2}] \\ &= F_0^{(2)}[X], \end{aligned}$$

where the last equality holds since $F_0^{(2)}[e^{-r_1-r_2}] = P_0^{(2)}[1] = Z_2$. In particular, the forward contract is a better deal than the futures contract.

Problem 2

We have $1 = P_0^{(2)}[e^{0.03+r}] = e^{-2 \cdot 0.04}\,\mathrm{E}^{(2)}[e^{0.03+0.03+\sigma Z}]$. Since $\mathrm{E}^{(2)}[e^{\sigma Z}] = e^{\sigma^2/2}$, this means that $0.08 = 0.06 + \sigma^2/2$ which in turn implies that $\sigma = 0.2$.

Problem 3

The first tree in Table 1 is the tree of forward prices, where G_0 is the current forward price for delivery of the share in one year and $u = 1 + \tanh(0.1) = 1.099668$ and $d = 1 - \tanh(0.1) = 0.900332$. The second tree is that tree with numerical values inserted. The third tree is the corresponding tree of spot prices, where $\gamma = \exp\{r\Delta t\} = 1.012578$. The fourth tree is the same tree with numerical values inserted. The fifth tree is the tree of European put option prices (no necessary here). Hence, the European put option price is 7.536457. The sixth tree is the tree of American put option prices. Hence, the American put option price is 7.85146.

Problem 4

The forward price is

$$G_0 = G_0^{(1)}[S_1] = e^{0.05}(100 - 5e^{-0.05/3}) \approx 99.95763.$$

Assuming Black's model gives that, under the forward probability distribution,

$$S_1 = G_0 e^{-\sigma^2/2 + \sigma Z}, \quad Z \sim N(0,1),$$

where $\sigma = 0.2$ by assumption. The derivative price is

$$\begin{aligned} P_0^{(1)}[S_1^3] &= e^{-0.05}\,\mathrm{E}^{(1)}[S_1^3] \\ &= e^{-0.05}\,\mathrm{E}^{(1)}[G_0^3 e^{-3\sigma^2/2 + 3\sigma Z}] \\ &= e^{-0.05} G_0^3 e^{-3\sigma^2/2} e^{9\sigma^2/2} \\ &= e^{-0.05} G_0^3 e^{3\sigma^2} \\ &\approx 1'071'145. \end{aligned}$$

0	1	2	3
G_0	G_0u	G_0u^2	G_0u^3
	G_0d	G_0du	G_0du^2
		G_0d^2	G_0d^2u
			G_0d^3

0	1	2	3
103.8212	114.16885	125.54783	138.06093
	93.47355	102.78987	113.03473
		84.15723	92.54501
			75.76945

0	1	2	3
S_0	$S_0u\gamma$	$S_0u^2\gamma^2$	$S_0u^3\gamma^3$
	$S_0d\gamma$	$S_0du\gamma^2$	$S_0du^2\gamma^3$
		$S_0d^2\gamma^2$	$S_0d^2u\gamma^3$
			$S_0d^3\gamma^3$

0	1	2	3
100	111.35001	123.98825	138.06093
	91.16568	101.51299	113.03473
		83.11181	92.54501
			75.76945

0	1	2	3
7.536457	2.793041	0	0
	12.469468	5.656347	0
		19.596281	11.45499
			28.23055

0	1	2	3
7.85146	2.793041	0	0
	13.107398	5.656347	0
		20.888190	11.45499
			28.23055

Table 1: Put option prices.

Problem 5

Let X be the unknown quantity, let S be the spot price in April 2011, let F be the futures price in April 2011, let F_0 be the current futures price, let $k = 15'000$, and let n be the position in the futures contract. The effective price is

$$S^e = XS - nk(F - F_0).$$

Write $\widetilde{S} = XS$, $\widetilde{F} = kF$, and $\widetilde{F}_0 = kF_0$. Then $S^e = \widetilde{S} - n(\widetilde{F} - \widetilde{F}_0)$ and we know that the choice of n that minimizes the variance of S^e is

$$n = \frac{\mathrm{Cov}(\widetilde{S}, \widetilde{F})}{\mathrm{Var}(\widetilde{F})}.$$

Since $F = S$ we have

$$\mathrm{Cov}(\widetilde{S}, \widetilde{F}) = k\,\mathrm{E}[XS^2] - k\,\mathrm{E}[XS]\,\mathrm{E}[S] = 15'000(\mathrm{E}[XS^2] - 80\,\mathrm{E}[XS]).$$

In particular, we have

$$n = \frac{\mathrm{E}[XS^2] - 80\,\mathrm{E}[XS]}{15'000 \cdot 400} = \frac{\mathrm{E}[XS^2] - 80\,\mathrm{E}[XS]}{60'000}$$

Under the assumption that $\mathrm{Cov}(X, S) = 0$ we have $\mathrm{E}[XS^2] - 80\,\mathrm{E}[XS] = \mathrm{E}[X]\,\mathrm{Var}(S) = 150'000 \cdot 400$ which gives

$$n = \frac{\mathrm{E}[XS^2] - 80\,\mathrm{E}[XS]}{15'000 \cdot 400} = \frac{150'000 \cdot 400}{15'000 \cdot 400} = 10.$$

Problem 6

Let X_t be the price in Euro of one Litas and let S_t be the price in Litas of one share of the stock. This means that $X_0 = 1/3.5$ and $S_0 = 100$. Consider the strategy of buying one share now (with Euros exchanged to Litas) collecting the dividend (and exchanging it to Euro) and finally selling the share (and exchanging the received Litas to Euro). This gives the cash flow $-X_0 S_0$ at time 0, $0.05 X_{2/3} S_{2/3}$ at time 2/3, and $X_1 S_1$ at time 1. By the law of one price,

$$X_0 S_0 = 0.05 P_0^{(2/3)}[X_{2/3} S_{2/3}] + P_0^{(1)}[X_1 S_1].$$

Considering the similar strategy of buying the share now but selling it at time 2/3 just before the dividend is paid, gives, from the law of one price, $X_0 S_0 = P_0^{(2/3)}[X_{2/3} S_{2/3}]$. Therefore the forward price in Euro is

$$G_0^{(1)}[X_1 S_1] = e^{0.03} P_0^{(1)}[X_1 S_1] = e^{0.03} 0.95 X_0 S_0 \approx 27.96948.$$