KTH Mathematics

Exam in SF2701 Financial Mathematics.
Tuesday June 3 2014 8.00-13.00.

Examiner: Camilla Landén, tel 070-719 3938.

Aids: Calculator.

General instructions: The solutions should be legible, easy to follow and not lacking in explanations. In particular, all notation used should be explained.

N.B. Number the pages and write your name on each sheet. Write only one exercise per sheet and state the number of the exercise on the sheet.

To pass the exam you need 25 points. A chance to take a supplementary examination is given if you have 23 or 24 points (to take the supplementary examination you must contact Camilla Landén within two weeks from the posting of the results of the exam).

The Black-Scholes formula and a table of the cumulative distribution function of a standard normal can be found at the end of the exam.

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1. (a) A stock price is currently \$40. It is known that at the end of three months it will be either \$43 or \$37. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a three-month European call option with a strike of \$40? ....(5p)

(b) Consider the standard Black-Scholes setting. Let K_1 and K_2 be given real numbers such that $0 \leq K_1 < K_2$. Now consider the portfolio corresponding to

- buying a call with strike price K_1 and selling a call with strike price K_2
- buying a put with strike price K_2 and selling a put with strike price K_1.

All options are European, written on the stock, and have the same expiry date T. A portfolio strategy like this is known as a *box spread*.

Determine the arbitrage price of the *box spread strategy* described above. (5p)

2. (a) Compute the price of an American put option written on a stock which will pay a dividend of 5% (discretely) of the stock value in three months time. The current stock price is \$100 and the volatility of the stock price is 25%. The maturity of the option is in six months and the strike price is \$100. The risk free interest rate with continuous compounding is 3% per annum. You should use a two period binomial model to price the option. .... (8p)

Hint: Think about whether you would exercise just before or just after dividend payment if you owned the option.

(b) If the underlying stock were not to pay a dividend before the maturity of the option would this increase or decrease the value of the put option? Please motivate! .... (2p)

3. In this exercise all interest rates are quoted per annum with continuous compounding.

(a) Suppose that the spot price of the Canadian dollar is U.S. \$0.99 and that the U.S. dollar/ Canadian dollar exchange rate has a volatility of 10% per annum. The risk-free interest rates in Canada and the United States are 4% and 5%, respectively.

i. Compute the forward exchange rate in nine months expressed as U.S. dollars/Canadian dollar (this is the same as the forward price of the exchange rate). .... (3p)

ii. Compute the price of a European put option to sell one Canadian dollar for U.S. \$1 in nine months. .... (4p)

(b) Consider a six-month European call option on the spot price of crude oil, that is an option to buy one barrel of crude oil in six months. The strike price is \$103 the current spot price of crude oil is \$102 per barrel, the six-month futures price of one barrel of crude oil is \$100, the risk-free interest rate is 5% per annum, and the volatility of the spot price is 15%, whereas the volatility of the futures price is 16.5%. Compute the price of the European call option. .... (3p)

4. (a) Suppose that the following bonds are currently trading in the market:

- A zero coupon bond with face value 100 and maturity six months trades at 98.51
- A zero coupon bond with face value 100 and maturity one year trades at 96.56
- A coupon bond with face value 100, maturity in two years, and a coupon of 2% per annum paid annually trades at 96.0874
- A coupon bond with face value 100, maturity in three years, and a coupon of 3% per annum paid annually trades at 97.0168

Compute the current term structure, i.e. the zero rates for six-months, one year, two years, and three years. Quote the rates per annum with continuous compounding. .... (4p)

(b) Given the term structure in (a), compute the swap rate of a two year fixed-for-floating swap, where six-month LIBOR is to be exchanged for the fixed swap rate R_s. Floating rate payments are to be made semi-annually, whereas the fixed payments are to be made once a year. The notational principal of the swap is one million dollars. The swap rate R_s should be quoted per annum using annual compounding. . . . . (3p)

(c) Now suppose for simplicity that the current term structure is flat, so the zero rates are the same for all maturities and equal to 4% per annum with continuous compounding. A one million dollar interest rate swap has a remaining life of nine months. Under the terms of the swap, six-month LIBOR is exchanged for 3% per annum compounded annually (floating rate payments are made semi-annually, and fixed rate payments are made annually). Three months ago the LIBOR rate was 3.5% per annum (compounded semi-annually). What is the current value of the swap to the party paying floating? . . . . (3p)

5. (a) Consider a contract written on the UK FTSE 100 index with price process S. The contract has a guaranteed minimum payout. More precisely, consider a five year contract which pays out 90% times the ratio of the terminal and the initial values of the FTSE, or 105% if otherwise it would be less. The claim X is thus given by

$$X = \max\{1.05, 0.9\tilde{S}_T\},$$

where T is 5 years and $\tilde{S}_T = S_T/S_0$.

The FTSE (Financial Times/Stock Exchange) 100 index is an index containing the largest 100 companies (by market capitalization) listed on the London Stock Exchange. As the FTSE 100 index (often referred to as the "Footsie") is composed of 100 different stocks their separate dividend payments will approximate a continuously paying stream. The following data are given

$$\begin{array}{rcll} \text{FTSE drift} & \mu & = & 7\% \\ \text{FTSE volatility} & \sigma & = & 15\% \\ \text{FTSE dividend yield} & \delta & = & 4\% \\ \text{UK interest rate} & r & = & 3.5\% \end{array}$$

Here the drift μ is the local mean rate of return of the index under P, everything is quoted per annum and the interest rate is quoted using continuous compounding.

Determine today's arbitrage price of the contract X. (5p)

Exercise (b) on next page.

(b) Consider the standard Black-Scholes setting. By a digital or binary option we mean a contract whose payoff depends in a discontinuous way on the terminal price of the underlying asset. Simple examples of binary options are *cash-or-nothing* options. The payoff at maturity of a cash-or-nothing call is

$$\mathrm{BCC}_T = \begin{cases} K & \text{if } S_T > K, \\ 0 & \text{otherwise} \end{cases}$$

where K denotes a prespecified amount of cash.

Your task is to determine the arbitrage free price of a binary cash-or-nothing call. The answer is allowed to be expressed in terms of the cumulative distribution function of a normally distributed random variable with expectation zero and variance one and the parameters of the problem (K, μ, σ, r, T, S_0). (5p)

Good luck!

Hints:
You are free to use the following in any of the above exercises.

- The density function of a normally distributed random variable with expectation m and variance σ^2 is given by

$$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/(2\sigma^2)}.$$

- Let Φ denote the cumulative distribution function for the $\mathrm{N}(0,1)$ distribution. Then

$$\Phi(-x) = 1 - \Phi(x).$$

- The standard Black-Scholes formula for the price $\Pi(t)$ of a European call option with strike price K and time of maturity T is $\Pi(t) = F(t, S(t))$, where

$$F(t,s) = s\Phi[d_1(t,s)] - e^{-r(T-t)} K\Phi[d_2(t,s)].$$

  Here Φ is the cumulative distribution function for the $N(0,1)$ distribution and

$$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}} \left\{ \ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t) \right\}, \\ d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}. \end{aligned}$$

The table on the next page shows values of $\Phi(x) = P(X \le x)$ for $X \in \mathrm{N}(0,1)$, i.e. Φ is the cumulative distribution function of the standard normal distribution. For negative values of x use that

$$\Phi(-x) = 1 - \Phi(x).$$