KTH Mathematics

Exam in SF2701 Financial Mathematics.
Monday June 8 2015 8.00-13.00.

Examiner: Camilla Landén, tel 070-719 3938.

Aids: Calculator.

General instructions: The solutions should be legible, easy to follow and not lacking in explanations. In particular, all notation used should be explained.

N.B. Number the pages and write your name on each sheet. Write only one exercise per sheet and state the number of the exercise on the sheet.

To pass the exam you need 25 points. A chance to take a supplementary examination is given if you have 23 or 24 points (to take the supplementary examination you must contact Camilla Landén within two weeks from the posting of the results of the exam).

The Black-Scholes formula and a table of the cumulative distribution function of a standard normal can be found at the end of the exam.

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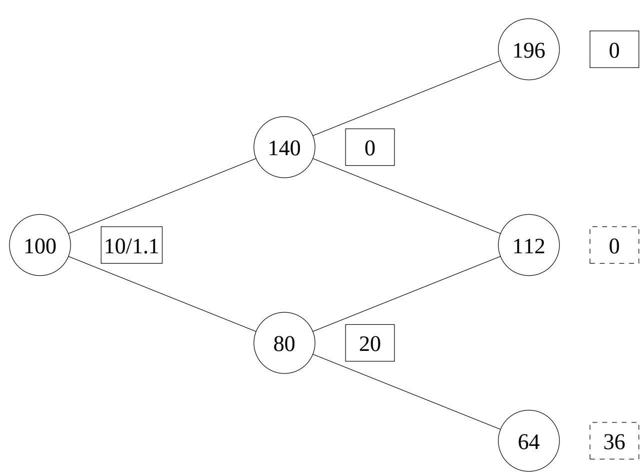
1. (a) In the binomial tree below the price of an *American put option* with strike price $K = 100$ kr and exercise date $T = 2$ years has been computed using the parameters $s_0 = 100$, $u = 1.4$, $d = 0.8$, $r = 10\%$, and $p = 0.75$. (The value of the stock is written in the nodes, and the value of the option is written in the adjacent boxes.)



Note that an early exercise of the option is optimal at the node with stock price 80 (this is why the last option values have been put in dashed boxes)!

Your task is to find the replicating portfolio for this option and to verify that the portfolio is self-financing. . . . . (5p)

(b) Consider the standard Black-Scholes setting. If a trader believes that the future price movement of the underlying stock will be large and more likely up than down, then buying a *strap option* could be of interest. This is an option strategy created by being long in one put and two call options (thus buying the three options). All options are European, written on the stock, with the exact same strike price, and the same expiry date.

Suppose that the current stock price is \$80, the strike price of the options used is also \$80, the expiry date is in six months, the volatility of the underlying stock is 30%, and the risk free interest rate with continuous compounding is 2% per annum.

Determine the arbitrage price of the *strap option* described above. . . . . . . (5p)

2. (a) Compute the price of an American call option written on a stock which will pay a dividend of 5% (discretely) of the stock value in two months time. The current stock price is \$100 and the volatility of the stock price is 20%. The maturity of the option is in four months and the strike price is \$100. The risk free interest rate with continuous compounding is 3% per annum. You should use a two period binomial model to price the option. (8p)

Hint: Think about whether you would exercise just before or just after dividend payment if you owned the option.

(b) Estimate the delta of the option after two months using your tree. Note that there will be two different values for delta, depending on where in the tree you end up. . . . . (2p)

3. In this exercise all interest rates are quoted per annum with continuous compounding.

(a) A European call option with strike price 350 SEK, and exercise date in six months, written on Hennes & Mauritz series B (HMB), currently trades at 13.77 SEK. The stock itself trades at 336 SEK. The risk free interest rate is 1% per annum.

i. Is the implied volatility for HMB 20%, or 30%? Please motivate your answer! . . . . (2p)

ii. Suppose that you have just sold the option described above. How many HMB stocks should you add to your portfolio in order to make it delta neutral? (If you do not know how to solve 3(a)i you can use a volatility of 25%.) . . . . (3p)

(b) Consider a six-month European put option written on the spot price of one ton of aluminum. The strike price of the option is \$2350, and the six-month futures price of one ton of aluminum is \$2332. The risk-free interest rate is 2% per annum, and the volatility of the futures price is 16.6%. Compute the price of the European put option. . . . . (5p)

4. (a) Suppose that the following bonds are currently trading in the market:

- A zero coupon bond with face value 100 and maturity six months trades at 99.00
- A zero coupon bond with face value 100 and maturity one year trades at 97.53
- A coupon bond with face value 100, maturity in two years, and a coupon of 2% per annum paid annually trades at 98.0106

Compute the current term structure, i.e. the zero rates for six-months, one year, and two years. Quote the rates per annum with continuous compounding. (3p)

(b) Consider the term structure in exercise 4a. Suppose two parties want to set up a forward rate agreement for the second year on \$1 million. What should they set the borrowing (lending) rate to? Quote the rate per annum with annual compounding. . . . . (3p)

(c) For all maturities the interest rate for the Swedish krona (SEK) is 2% per annum and the U.S. dollar (USD) interest rate is 4% per annum. The current exchange rate is 8.4 SEK/USD. In a currency swap agreement, a Swedish company pays 4% per annum in USD and receives 2% per annum in SEK. The principals in the two currencies are \$1 million USD and 8 million SEK and. Payments are exchanged every year, with one exchange just having been made. The swap will last two more years (thus two exchanges remain). All interest rates are quoted with annual compounding.

What is the value of the swap to the Swedish company? . . . . (4p)

5. For this exercise consider the standard Black-Scholes model.

(a) Consider a forward contract, with a particular forward price f, and delivery date T, written on the underlying stock. Show how to replicate the forward contract using put and call options. Make sure to state the relationships between the characteristics of the forward contract and the characteristics of the options. . . . . (5p)

Exercise (b) on next page.

(b) A *power option* is a contingent T-claim X defined by

$$X = \phi(S_T) = \max\{S_T^p - K, 0\},$$

for some real constants $K > 0$ and $p > 0$. Denote by $c^p(t, s, K, T, r, \sigma)$ the arbitrage price at time t of a power option with exercise price K and expiry date T, when the current price of the underlying is s, the interest rate is r, and the volatility of the underlying is σ.

Your task is to determine $c^p(0, s, K, T, r, \sigma)$. .... (5p)

Hint: "Brute force" works, but comparing the distribution of S^p to that of an underlying stock paying a continuous dividend yield of δ, and expressing the price in terms of the Black-Scholes formula for that case, will save you a lot of time.

Good luck!

Hints:

You are free to use the following in any of the above exercises.

- The density function of a normally distributed random variable with expectation m and variance σ^2 is given by

$$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/(2\sigma^2)}.$$

- Let Φ denote the cumulative distribution function for the $\mathrm{N}(0,1)$ distribution. Then

$$\Phi(-x) = 1 - \Phi(x).$$

- The standard Black-Scholes formula for the price $\Pi(t)$ of a European call option with strike price K and time of maturity T is $\Pi(t) = F(t, S(t))$, where

$$F(t, s) = s\Phi[d_1(t, s)] - e^{-r(T-t)} K\Phi[d_2(t, s)].$$

  Here Φ is the cumulative distribution function for the $N(0,1)$ distribution and

$$\begin{aligned} d_1(t, s) &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \\ d_2(t, s) &= d_1(t, s) - \sigma\sqrt{T-t}. \end{aligned}$$

The table on the next page shows values of $\Phi(x) = P(X \le x)$ for $X \in \mathrm{N}(0,1)$, i.e. Φ is the cumulative distribution function of the standard normal distribution. For negative values of x use that

$$\Phi(-x) = 1 - \Phi(x).$$