KTH Mathematics

Exam in SF2701 Financial Mathematics.
Friday June 3 2016 8.00-13.00.

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Aids: Calculator.

General instructions: The solutions should be legible, easy to follow and not lacking in explanations. In particular, all notation used should be explained.

N.B. Number the pages and write your name on each sheet. Write only one exercise per sheet and state the number of the exercise on the sheet.

To pass the exam you need 25 points. A chance to take a supplementary examination is given if you have 23 or 24 points (to take the supplementary examination you must contact Camilla Landén within two weeks from the posting of the results of the exam).

The Black-Scholes formula and a table of the cumulative distribution function of a standard normal can be found at the end of the exam.

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**1.** (a) A stock price is currently \$100. It is known that at the end of three months it will be either \$120 or \$80. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a three-month European put option with a strike of \$100? ............................................(5p)

(b) Consider the standard Black-Scholes setting. A conversion is an arbitrage strategy in options trading that can be performed for a riskless profit when options are overpriced relative to the underlying stock. To do a conversion, the trader buys the underlying stock and offsets it with a synthetic short stock, i.e by buying a put option and selling a call option (the conversion thus consists of: long put + short call + long underlying). Both options are European, written on the stock, with the same strike price, and the same expiry date.

Suppose that the current stock price is \$80, the strike price of the options used is \$85, and the expiry date is in six months. Furthermore suppose that the call option currently trades at \$5.0 and the put option at \$9.3 and that the risk free interest rate with continuous compounding is 2% per annum.

i. What is the arbitrage price of the synthetic short stock, i.e. the long put and short call position? ............................................(3p)
ii. What profit do you lock in if you implement the conversion? ........(2p)

**2.** (a) Compute the price of an American put option written on a the futures price of a commodity. The current futures price is \$100 and the volatility of the futures price is 25%. The maturity of the option is in six months and the strike price is \$100. The risk free interest rate with continuous compounding is 2% per annum. You should use a two period binomial model to price the option. (8p)

(b) Estimate the current value of the delta of the option (note that the underlying of the option is the futures price, not the stock itself). ..................(2p)

**3.** In this exercise all interest rates are quoted per annum with continuous compounding.

(a) Consider a nine-month European call option written on a stock index paying a continuous dividend yield of 5%. The value of the stock index is currently 100, the strike price of the option is 97, the risk-free interest rate is 3% per annum, and the volatility of the index is 25%. Compute the price of the European call option. ............................................................ (3p)

(b) Compute the delta of the call option. .................................. (3p)
**Hint:** You have to think a bit to incorporate the yield. Using the definition of delta, delta for a non-dividend paying stock, and the chain rule should do the trick.

(c) Now consider another stock index also paying a continuous dividend yield, $q$, which at this point is unknown to you. The current value of the index is 512 and the price of a European call option with strike 505 and maturity in three months is 32, and the price of a European put option with strike 505 and maturity in three months is 28. The risk-free interest rate is 3% per annum.
Compute the continuous dividend yield, $q$. .............................(4p)

**4.** (a) Suppose that the current term structure looks as follows:

| $T$ | $r(0,T)$ |
|---|---|
| 0.5 | 2.0% |
| 1.0 | 2.0% |
| 1.5 | 2.5% |
| 2.0 | 3.0% |
| 2.5 | 3.5% |
| 3.0 | 3.75% |

Here the zero rates are quoted per annum with continuous compounding.

Compute the prices for the following bonds:

i. A zero coupon bond with face value 100 and maturity six months,
ii. A coupon bond with face value 100, maturity in two years, and a coupon of 2% per annum paid semi-annually,
iii. A coupon bond with face value 100, maturity in three years, and a coupon of 3% per annum paid annually.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (3p)

(b) In a fixed-for-floating currency swap agreement Sterling LIBOR on a principal of £7 million is paid and 3% on a principal of $10 million is received with payments being made annually (just as for a fixed-for-fixed currency swap there will be a final exchange of principal in the same direction as the interest payments at the end of the swaps life).

Assume that the British pound term structure (£ or GBP) is given by the term structure found in exercise 4a, and that the U.S. dollar term structure ($ or USD) is flat and 4% per annum with annual compounding. Furthermore assume that the current exchange rate is 0.69 GBP/USD, and that the remaining life of the swap is two years (with one payment just having been made).

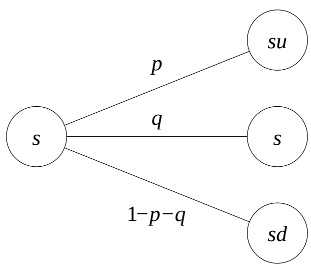
What is the value of the swap in GBP to the party paying British pounds and receiving dollars? . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (7p)

**Hint:** One way of computing the value of the fixed-for-floating currency swap is to say that the company holds a portfolio consisting of

- a fixed-for-fixed currency swap where 3% on a principal of $10 million is received and (say) 4% on a principal of £7 million is paid, and
- a fixed-for-floating interest rate swap where 4% is received and LIBOR is paid on a principal of £7 million.

You do not have to see it this way, and if you prefer, you can compute the value some other way.

**5.** (a) Consider a one period model very similar to the one period binomial model, the only difference being that the stock price $S$ can also stay the same with a certain probability $q$, as depicted in the figure below.




Given that $d < 1 + r < u$ is the model complete or not? A proof or counterexample is required (you may use the first and second fundamental theorem without proving them). . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (5p)

(b) For this exercise consider the standard Black-Scholes model. Let $c(s,T,K,\sigma)$ denote the price of a European call option when the spot price is $s$, time to maturity $T$, strike price $K$, and volatility $\sigma$. That is,

$$C(t,S(t)) = c(S(t), T-t, K, \sigma)$$

is the price at time $t$ of a European call option with exercise time $T$ and strike price $K$. Consider a call option on an at-the-money call option, i.e. a claim $X$ with maturity $T_0 < T$ and strike $K_0$, that gives the holder the option to, at time $T_0$ and at price $K_0$, buy a European call option with exercise time $T$ and strike price $S(T_0)$. The payoff of $X$ at time $T_0$ is thus given by

$$X = \max\{c(S(T_0), T-T_0, S(T_0), \sigma) - K_0, 0\}.$$

i. Determine the price of $X$ today. Your answer may contain density functions and cumulative distribution functions of known distributions. ....... (4p)

ii. Compute the price numerically when $r = 0.03$, $\sigma = 0.2$, $s_0 = 100$, $T = 1$, $T_0 = 0.5$, and $K_0 = c(s_0, T, s_0, \sigma)$ (an at-the-money call option on an at-the-money call option!). ........................................... (1p)

*Good luck!*

**Hints:**

You are free to use the following in any of the above exercises.

- The density function of a normally distributed random variable with expectation $m$ and variance $\sigma^2$ is given by

$$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/(2\sigma^2)}.$$

- Let $\Phi$ denote the cumulative distribution function for the $\mathrm{N}(0,1)$ distribution. Then

$$\Phi(-x) = 1 - \Phi(x).$$

- The standard Black-Scholes formula for the price $\Pi(t)$ of a European call option with strike price $K$ and time of maturity $T$ is $\Pi(t) = C(t, S(t))$, where

$$C(t,s) = s\Phi[d_1(t,s)] - e^{-r(T-t)} K \Phi[d_2(t,s)].$$

Here $\Phi$ is the cumulative distribution function for the $N(0,1)$ distribution and

$$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}} \left\{ \ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t) \right\}, \\ d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}. \end{aligned}$$

The table on the next page shows values of $\Phi(x) = P(X \le x)$ for $X \in \mathrm{N}(0,1)$, i.e. $\Phi$ is the cumulative distribution function of the standard normal distribution. For negative values of $x$ use that

$$\Phi(-x) = 1 - \Phi(x).$$