KTH Mathematics

Exam in SF2701 Financial Mathematics.
Wednesday August 17 2016 8.00-13.00.

Examiner: Camilla Landén, tel 070-719 3938.

Aids: Calculator.

General instructions: The solutions should be legible, easy to follow and not lacking in explanations. In particular, all notation used should be explained.

N.B. Number the pages and write your name on each sheet. Write only one exercise per sheet and state the number of the exercise on the sheet.

To pass the exam you need 25 points. A chance to take a supplementary examination is given if you have 23 or 24 points (to take the supplementary examination you must contact Camilla Landén within two weeks from the posting of the results of the exam).

The Black-Scholes formula and a table of the cumulative distribution function of a standard normal can be found at the end of the exam.

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1. (a) A stock price is currently \$100. It is known that at the end of three months it will be either \$115 or \$75. The risk-free interest rate is 5% per annum with continuous compounding. Compute the replicating portfolio for a three-month European call option with a strike of \$97? .... (5p)

(b) Consider the standard Black-Scholes setting.

A *married put*, or *protective put*, is a portfolio strategy where an investor buys a certain number of shares of a stock and, at the same time, the same number of put options on the same stock.

i. Draw the payoff function of a married put, given that it only involves one share of the stock, and consequently only one put option, and that the strike price of the put option is K. .... (2p)

ii. Compute the arbitrage price of the married put given that the current stock price is \$90, the strike price of the put option used is \$85, and the expiry date is in three months. Furthermore the volatility of the stock is 30%, and the risk free interest rate with continuous compounding is 2% per annum. .... (3p)

2. (a) Compute the price of an American call option written on a stock which will pay a dividend of \$5 in three months time. The current stock price is \$100 and the volatility of the stock price is 35%. The maturity of the option is in six months and the strike price is \$100. The risk free interest rate with continuous compounding is 3% per annum. You should use a two period binomial model to price the option. . . . . (8p)

Hint: Think about whether you would exercise just before or just after dividend payment if you owned the option.

(b) If the underlying stock were not to pay a dividend before the maturity of the option would this increase or decrease the value of the call option? Please motivate! . . . . (2p)

3. In this exercise all interest rates are quoted per annum with continuous compounding, and are considered to be deterministic.

(a) A stock is expected to pay a dividend of \$5 of in three months time. The stock price is \$100, and the risk-free rate of interest is 2% per annum with continuous compounding. An investor has just taken a long position in a six-month futures contract on the stock.

i. What are the futures price and the initial value of the futures contract? . . . . (3p)

ii. Consider a three-month European call option written on the futures price. The strike price of the option is 96, and the volatility of the futures price is 30%. Compute the price of the futures option. (3p)

(b) In the standard Black-Scholes setting, compute the delta of a portfolio consisting of a long position in a European call option on the stock, with strike price K and exercise date T, and a short position in a European put option with the same strike price, exercise date, and underlying as the call option. . . . . . . (4p)

4. (a) Suppose that the following bonds are currently trading in the market:

- A zero coupon bond with face value 100 and maturity in six months trades at 99.50
- A coupon bond with face value 100, maturity in 18 months, and a coupon of 3% per annum paid annually trades at 103.693
- A coupon bond with face value 100, maturity in 18 months, and a coupon of 2% per annum paid semi-annually trades at 100.7326
- A coupon bond with face value 100, maturity in two years, and a coupon of 2% per annum paid annually trades at 99.9707

i. Compute the current term structure, i.e. the zero rates for six months, one year, 18 months, and two years. Quote the rates per annum with continuous compounding. . . . . (3p)

ii. Compute the forward rate for the one-year period beginning in one year. Quote the rate per annum with continuous compounding. . . . .(3p)

(b) Using the term structure in exercise 4a, value a forward rate agreement where you will pay 2.6% (compounded annually) for the one-year period beginning in one year on \$1 million. .... (4p)

5. For this exercise consider the standard Black-Scholes model.

(a) Consider a T-claim X which gives the holder of the claim $1/S(T_0)$ stocks at time T, where T_0 is a fixed time such that $T_0 < T$. This means that at time T the holder of the claim will receive

$$X = \frac{S(T)}{S(T_0)}.$$

The price for $t \in [0, T_0]$ is $e^{-r(T_0-t)}$.

Construct a replicating portfolio for the claim just described. .... (5p)

(b) A *pay later option* written on a stock is financial contract stipulating that the buyer of the pay later option has the obligation to exercise the option when it is in the money and to pay the premium. This means that as soon as the difference between the price of the underlying and the strike price is positive exercise takes place, regardless of how big the difference is, i.e. the amount by which the option is in the money. The payoff at the exercise date T of a pay later option with strike price K written on a stock with price process S is thus given by $Y = \phi(S_T)$, where the function ϕ is defined by

$$\phi(x) = \begin{cases} x - K - p, & \text{if } x > K, \\ 0, & \text{otherwise.} \end{cases}$$

Here p denotes the premium which the holder of the option must pay upon exercise. The premium p is determined when the contract is initiated (at time $t = 0$) in such a way that the current price, $\Pi(0, Y)$, of the contract is zero, i.e. $\Pi(0, Y) = 0$.

Your task is to compute the correct premium p for the pay later option described above, when the current price of the underlying stock is s, the interest rate is r, and the volatility of the underlying stock is σ. .... (5p)

Hint: It may help to think of the payoff as

$$\phi(x) = \begin{cases} x - K, & \text{if } x > K, \\ 0, & \text{otherwise.} \end{cases} - \begin{cases} p, & \text{if } x > K, \\ 0, & \text{otherwise.} \end{cases}$$

Good luck!

Hints:
You are free to use the following in any of the above exercises.

- The density function of a normally distributed random variable with expectation m and variance σ^2 is given by

$$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/(2\sigma^2)}.$$

- Let Φ denote the cumulative distribution function for the $\mathrm{N}(0,1)$ distribution. Then

$$\Phi(-x) = 1 - \Phi(x).$$

- The standard Black-Scholes formula for the price $\Pi(t)$ of a European call option with strike price K and time of maturity T is $\Pi(t) = C(t, S(t))$, where

$$C(t,s) = s\Phi[d_1(t,s)] - e^{-r(T-t)} K \Phi[d_2(t,s)].$$

  Here Φ is the cumulative distribution function for the $N(0,1)$ distribution and

$$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}} \left\{ \ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t) \right\}, \\ d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}. \end{aligned}$$

The table on the next page shows values of $\Phi(x) = P(X \le x)$ for $X \in \mathrm{N}(0,1)$, i.e. Φ is the cumulative distribution function of the standard normal distribution. For negative values of x use that

$$\Phi(-x) = 1 - \Phi(x).$$