KTH Mathematics

Exam in SF2701 Financial Mathematics.
Wednesday August 16 2017 8.00-13.00.

Examiner: Camilla Landén, tel 070-719 3938.

Aids: Calculator.

General instructions: The solutions should be legible, easy to follow and not lacking in explanations. In particular, all notation used should be explained.

N.B. Number the pages and write your name on each sheet. Write only one exercise per sheet and state the number of the exercise on the sheet.

To pass the exam you need 25 points. A chance to take a supplementary examination is given if you have 23 or 24 points (to take the supplementary examination you must contact Camilla Landén within two weeks from the posting of the results of the exam).

The Black-Scholes formula and a table of the cumulative distribution function of a standard normal can be found at the end of the exam.

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1. (a) State the first fundamental theorem of arbitrage theory. ....(5p)

(b) A compound option is an option written on an option. The payoff function of a European call on a put with strike K and maturity T is

$$X_{callonput} = \max\{P(T, S_T) - K, 0\},$$

where $P(T, S_T)$ denotes the price of the underlying put option at time T (the maturity of this option is $T_1 > T$ and the strike is K_1). Similarly the payoff function of a European put on a put with strike K and maturity T is

$$X_{putonput} = \max\{K - P(T, S_T), 0\}.$$

What put-call parity relationship exists between the price of a European call on a put and a European put on the same put? ....(5p)

2. (a) Compute the price of an American call option written on a stock which will pay a dividend of 6% (discretely) of the stock value in four months time. The current stock price is \$50 and the volatility of the stock price is 25%. The maturity of the option is in eight months and the strike price is 50\$. The risk free interest rate with continuous compounding is 4% per annum. You should use a two period binomial model to price the option. .... (8p)

Hint: Think about whether you would exercise just before or just after dividend payment if you owned the option.

(b) If the underlying stock were not to pay a dividend before the maturity of the option would this increase or decrease the value of the call option? Please motivate! .... (2p)

3. In this exercise all interest rates are quoted per annum with continuous compounding.

(a) An index paying a continuous dividend yield of 4% currently trades at \$100. The risk-free rate of interest is 2% per annum with continuous compounding. An investor has just taken a long position in a six-month futures contract on the index.

i. What are the futures price and the initial value of the futures contract? .... (3p)

ii. Consider a six-month European call option written on the stock. The strike price of the option is 95, and the volatility of the stock is 30%. Compute the price of the European call option. .... (3p)

Hint: The volatility of the futures price is the same as the volatility of the stock price.

(b) In the standard Black-Scholes setting, compute the delta of a portfolio consisting of a long position in a European call option on the stock, with strike price K and exercise date T, and a long position in a European put option with the same strike price, exercise date, and underlying as the call option. Your answer may contain density functions and cumulative distribution functions of known distributions and the parameters of the problem: μ (local mean rate of return of the stock under P), σ, r, T, S_0, and K. .... (4p)

4. (a) Suppose the current term structure looks as follows:

T	$r(0,T)$
0.5	1.7%
1	2.0%
1.5	2.3%
2	2.5%

Here $r(0,T)$ denotes the current zero rate for the time period T.

Compute the prices of

i. A zero coupon bond with face value 100 and maturity six months,
ii. a zero coupon bond with face value 100 and maturity one year,
iii. a coupon bond with face value 100, maturity in two years, and a coupon of 2% per annum paid semi-annually.

.... (3p)

(b) Using the term structure in exercise 4a, value a forward rate agreement where you will pay 3% (compounded annually) for the second year on \$1 million. .... (3p)

(c) Suppose that on the market a one year and a two year interest rate swap have just been set up. Under the terms of the one year swap 1.7% per annum (compounded annually) is exchanged for one-year LIBOR on a principal of \$1.000.000 (payments are thus made yearly). Under the terms of the two year swap 2.0% per annum (compounded annually) is exchanged for one-year LIBOR on a principal of \$1.000.000 (payments are thus made yearly).

Determine the one-year and two-year zero rates from the given swap rates (forget about the term structure given in exercise 4a). .... (4p)

5. (a) Asian options are options where the payoff depends on the arithmetic average of the price

$$S_T^{ave} = \frac{1}{T}\int_0^T S_t dt$$

of the underlying asset during the life time of the option.

The payoff from an *average price call* with strike price K and maturity T is

$$X_{ave\_price\_call} = \max\{S_T^{ave} - K, 0\}.$$

Similarly, the payoff from an *average strike call* with maturity T is

$$X_{ave\_strike\_call} = \max\{S_T - S_T^{ave}, 0\}.$$

The payoffs for an *average price put* with strike price K and maturity T, and an *average strike put* with maturity T are

$$X_{ave\_price\_put} = \max\{K - S_T^{ave}, 0\}, \text{ and } \quad X_{ave\_strike\_put} = \max\{S_T^{ave} - S_T, 0\}$$

respectively.

Now let C_{ave_price}, and C_{ave_strike} denote the prices today of the average price call and average strike call, described above. Similarly, let P_{ave_price}, and P_{ave_strike} denote the prices today of the average price put and average strike put, respectively. Finally, let C and P denote the prices of a standard European call and put option with strike price K and maturity T.

Show that

$$C_{ave\_price} + C_{ave\_strike} - C = P_{ave\_price} + P_{ave\_strike} - P.$$

.... (5p)

Hint: There are six ways to order K, S_T, and S_T^{ave}.

(b) For this exercise consider the standard Black-Scholes model. A *cash-or-nothing put* is a T-claim X, such that $X = \phi(S_T)$, where the payoff function ϕ is defined by

$$\phi(s) = \begin{cases} K_1, & \text{if } s < K_2, \\ 0, & \text{otherwise.} \end{cases}$$

Here K_1 and K_2 are non-negative constants.

Determine the price of X today. Your answer may contain density functions and cumulative distribution functions of known distributions and the parameters of the problem: μ (local mean rate of return of the stock under P), σ, r, T, S_0, K_1, and K_2. .... (5p)

Good luck!

Hints:

You are free to use the following in any of the above exercises.

- The density function of a normally distributed random variable with expectation m and variance σ^2 is given by

$$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/(2\sigma^2)}.$$

- Let Φ denote the cumulative distribution function for the N$(0,1)$ distribution. Then

$$\Phi(-x) = 1 - \Phi(x).$$

- The standard Black-Scholes formula for the price $\Pi(t)$ of a European call option with strike price K and time of maturity T is $\Pi(t) = C(t, S(t))$, where

$$C(t,s) = s\Phi[d_1(t,s)] - e^{-r(T-t)}K\Phi[d_2(t,s)].$$

Here Φ is the cumulative distribution function for the $N(0,1)$ distribution and

$$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \\ d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}. \end{aligned}$$

The table on the next page shows values of $\Phi(x) = P(X \leq x)$ for $X \in$ N$(0,1)$, i.e. Φ is the cumulative distribution function of the standard normal distribution. For negative values of x use that

$$\Phi(-x) = 1 - \Phi(x).$$