Avd. Matematisk statistik

KTH Matematik

TENTAMEN I SF2940 SANNOLIKHETSTEORI/EXAM IN SF2940 PROBABILITY THEORY SATURDAY THE 11^{th} OF FEBRUARY 2012 2.00 p.m.–7.00 p.m.

Examinator: Timo Koski, tel. 790 71 34, email: tjtkoski@kth.se

Tillåtna hjälpmedel Means of assistance permitted: Appendix 2 in A.Gut: An Intermediate Course in Probability. Formulas for probability theory SF2940. Pocket calculator.

You should define and explain your notation. Your computations and your line of reasoning should be written down so that they are easy to follow. Numerical values should be given with the precision of two decimal points. You may apply results stated in a part of an exam question to another part of the exam question even if you have not solved the first part. The number of exam questions (Uppgift) is six (6).
Solutions written in Swedish are, of course, welcome.

Each question gives maximum ten (10) points. 30 points will guarantee a passing result. The grade Fx (the exam can completed by extra examination) for those with 27−29 points.

Solutions to the exam questions will be available at
`http://www.math.kth.se/matstat/gru/sf2940/`
starting from Saturday 11^{th} of February 2012 at 7.30 p.m..

The exam results will be announced at the latest on Friday the 24^{th} of February 2012.

Your exam paper will be retainable at elevexpeditionen during a period of seven weeks after the date of the exam.

LYCKA TILL!

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Uppgift 1

The bivariate random variable (X, Y) has the joint probability density

$$f_{X,Y}(x, y) = 2xy\, e^{-x-y}, \quad y > x > 0.$$

a) Find the joint density of (U, V) given by

$$U = \frac{X}{X+Y}, \quad V = X + Y.$$

(6 p)

b) Show that U and V are independent and give their respective marginal densities. (4 p)

Uppgift 2

We consider two random variables X and Y such that $X \mid Y = y \in \mathrm{Po}(y)$ and $Y \in \mathrm{Po}(\lambda)$, $\lambda > 0$.

a) Show that the probability generating function $g_{X+Y}(t)$ of $X + Y$ is

$$g_{X+Y}(t) = e^{\lambda\left(te^{t-1}-1\right)}.$$

(8 p)

b) Find $P(X + Y = 0)$ and $P(X + Y = 1)$. Comment upon your findings in words ! (2 p)

Uppgift 3

Let $X_1, X_2, \ldots, X_n, \ldots$, be independent, identically distributed and positive (i.e. $P(X_i > 0) = 1$) random variables with the probability density $f_X(x)$. Assume that $f_X(x)$ has at 0 the limit from the right equal to $c > 0$, i.e., $\lim_{x \to 0} f_X(x) = c > 0$.

a) Let $Y_n = \min(X_1, X_2, \ldots, X_n)$. Show that Y_n converges in probability to 0, as $n \to \infty$,

$$Y_n \xrightarrow{P} 0.$$

(3 p)

b) Show now that

$$nY_n \xrightarrow{d} \mathrm{Exp}\left(\frac{1}{c}\right)$$

as $n \to \infty$. (7 p)

Uppgift 4

The stochastic process $\mathbf{X} = \{X(t) \mid -\infty < t < \infty\}$ is Gaussian and weakly stationary. Its mean function is $= 0$ and its autocorrelation function is with $h = t - s$ given by

$$R_{\mathbf{X}}(h) = E[X(t)X(s)] = \frac{1}{2}e^{-|h|}.$$

a) Find $P(X(t) \leq X(t-1) + 4.0)$. (9 p)

b) Find $P(X(t+1) \leq X(t) + 4.0)$. Please justify your answer. (1 p)

Uppgift 5

$\mathbf{W} = \{W(t) \mid t \geq 0\}$ is a Wiener process and $\mathbf{N} = \{N(t) \mid t \geq 0\}$ is a Poisson process with the intensity $\lambda > 0$. $\mathbf{N}$ is independent of $\mathbf{W}$. Let T be the time for the occurrence of the first event in $\mathbf{N}$.

a) Show that the characteristic function of $W(T)$ is

$$\varphi_{W(T)}(t) = \frac{1}{1 + \frac{t^2}{2\lambda}}.$$

(9 p)

b) What is the distribution of $W(T)$? (1 p)

Uppgift 6

$X_1, X_2, \ldots$ are independent and identically distributed random variables with $E[X_i] = \mu$ and $\text{Var}[X_i] = \sigma^2 < +\infty$. We set $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

a) Check that

$$\sum_{i=1}^{n} \left(X_i - \overline{X}_n\right)^2 = \sum_{i=1}^{n} (X_i - \mu)^2 - n\left(\overline{X}_n - \mu\right)^2$$

(1 p)

b) Let

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n} \left(X_i - \overline{X}_n\right)^2. \tag{1}$$

This is a well known estimator of variance in statistics. Show that

$$S_n^2 \xrightarrow{P} \sigma^2,$$

as $n \to +\infty$. (2 p)

c) When S_n^2 is given in (1), we have $S_n = \sqrt{S_n^2}$. Find the limiting distribution of

$$\frac{\sqrt{n}\left(\overline{X}_n - \mu\right)}{S_n},$$

as $n \to +\infty$. Justify your steps of solution carefully. (7 p)

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SOLUTIONS TO THE EXAM SATURDAY THE 11^{th} OF FEBRUARY 2012 02.00 p.m.– 07.00 p.m..

Uppgift 1

We write the density of (X, Y) as

$$f_{X,Y}(x, y) = 2xye^{-x-y}I(y > x > 0), \tag{2}$$

where $I(y > x > 0)$ is the indicator function of the set $y > x > 0$.

a) We write $u = x/(x + y), v = x + y$ for $x, y > 0$ and obtain therefore

$$x = uv, y = (1 - u)v \quad \text{for } 0 < u < 1, v > 0.$$

We have the Jacobian

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ -v & (1-u) \end{vmatrix} = v(1 - u) + vu = v.$$

The theorem about transformation of variables gives

$$f_{U,V}(u, v) = f_{X,Y}(uv, (1 - u)v)|J|$$

and from (2) and the expression for the Jacobian

$$= 2uv(1 - u)ve^{-v}I((1 - u)v > uv > 0, v > 0)v.$$

$$\text{ANSWER a): } \underline{f_{U,V}(u, v) = 2u(1 - u)v^3e^{-v}I((1 - u)v > uv > 0, v > 0).}$$

b) If U and V are to be independent, their joint density must be factorized for all (u, v). We observe that a property of indicator functions gives

$$I((1 - u)v > uv > 0, v > 0) = I((1 - u)v > uv > 0)I(v > 0)$$

and by elementary manipulation of inequalities

$$I((1 - u)v > uv > 0) = I(0 < u < 1/2).$$

Thus we conclude from a) that

$$f_{U,V}(u, v) = 2u(1 - u)I(0 < u < 1/2)v^3e^{-v}I(v > 0).$$

Hence we can write $f_{U,V}(u, v) = f_U(u)f_V(v)$ for all (u, v), where

$$f_U(u) = 12u(1 - u)I(0 < u < 1/2), \quad f_V(v) = (1/6)v^3e^{-v}I(v > 0).$$

where we recognize $f_V(v)$ as the density of $\Gamma(4,1)$.

ANSWER b): $\underline{f_U(u) = 12u(1-u)I(0<u<1/2), f_V(v) = (1/6)v^3e^{-v}I(v>0).}$

Uppgift 2

We consider two random variables X and Y such that $X \mid Y = y \in \text{Po}(y)$ and , $\lambda > 0$.

a) By definition the probability generating function $g_{X+Y}(t)$ is

$$g_{X+Y}(t) = E\left[t^{X+Y}\right]$$

and by double expectation

$$= E\left[E\left[t^{X+Y} \mid Y\right]\right]$$

and we take out what is known to get

$$= E\left[t^Y E\left[t^X \mid Y\right]\right].$$

From the Collection of Formulas, section 8.1.4. we get, since $X \mid Y = y \in \text{Po}(y)$, that $E\left[t^X \mid Y\right] = e^{Y(t-1)}$. Thus we have

$$= E\left[t^Y e^{Y(t-1)}\right] = E\left[\left(te^{(t-1)}\right)^Y\right].$$

But by definition of the probability generating function $g_Y(t)$ we have

$$= E\left[\left(te^{(t-1)}\right)^Y\right] = g_Y\left(te^{(t-1)}\right).$$

Since $Y \in \text{Po}(\lambda)$, the Collection of Formulas, section 8.1.4. yields $g_Y(t) = e^{\lambda(t-1)}$. Thus

$$g_{X+Y}(t) = g_Y\left(te^{(t-1)}\right) = e^{\lambda\left(te^{t-1}-1\right)},$$

as was to be shown.

b) In order to find $P(X+Y=0)$ we note that $X+Y=0$ is equivalent to $X=0, Y=0$. Then, of course, $X \mid Y = 0 \in \text{Po}(0)$, and $P(X=0 \mid Y=0) = 1$. Hence

$$P(X+Y=0) = P(X=0, Y=0) = P(X=0 \mid Y=0)\, P(Y=0) = P(Y=0) = e^{-\lambda},$$

since $Y \in \text{Po}(\lambda)$. By the workings of probability generating functions we can check this result by a) as

$$P(X+Y=0) = g_{X+Y}(0) = e^{-\lambda}.$$

For evaluation of $P(X+Y=1)$ we note that $X+Y=1$ is equivalent to $X=0, Y=1$ or $X=1, Y=0$. The case $X=1, Y=0$ is not possible, as we have shown that $X \mid Y = 0 \in$ Po(0), and $P(X=0 \mid Y=0) = 1$. Then

$$P(X+Y=1) = P(X=0, Y=1) = P(X=0 \mid Y=1)\, P(Y=1) = e^{-1} \cdot \lambda e^{-\lambda}.$$

Again, we can check this, since

$$P(X+Y=1) = \frac{d}{dt} g_{X+Y}(t) \mid_{t=0}$$

and
$$\frac{d}{dt} g_{X+Y}(t) = \lambda \left(e^{t-1} + te^{t-1}\right) e^{\lambda\left(te^{t-1}-1\right)}.$$
which again gives $e^{-1} \cdot \lambda e^{-\lambda}$ as the sought probability.

ANSWER b): $\underline{P(X+Y=0) = e^{-\lambda}, P(X+Y=1) = \lambda e^{-(\lambda+1)}}.$

Uppgift 3

a) We are in view of the very definition of convergence in probability expected to show that for any $\epsilon > 0$
$$P(\mid Y_n \mid > \epsilon) \to 0$$
as $n \to \infty$. Since the X_is are positive, $Y_n > 0$ and then
$$P(\mid Y_n \mid > \epsilon) = P(Y_n > \epsilon).$$

By construction
$$P(Y_n > \epsilon) = P(\min(X_1, X_2, \ldots, X_n) > \epsilon) = P\left(\cap_{i=1}^{n} \{X_i > \epsilon\}\right),$$
since $\min(X_1, X_2, \ldots, X_n) > \epsilon$ if and only if every $X_i > \epsilon$. Since $X_1, X_2, \ldots, X_n, \ldots$ are independent, identically distributed we get
$$P\left(\cap_{i=1}^{n} \{X_i > \epsilon\}\right) = \prod_{i=1}^{n} P(X_i > \epsilon) = \prod_{i=1}^{n} (1 - F_X(\epsilon)) = (1 - F_X(\epsilon))^n$$
where $F_X(\epsilon)$ is the value of the common distribution function at ϵ. Note that if there was a number $a > 0$ that $F_X(a) = 1$, then $P(Y_n > a) = 0$. Since $\lim_{x \to 0} f_X(x) = c > 0$, we know that $F_X(\epsilon) > 0$ for any ϵ. Therefore $0 < 1 - F_X(\epsilon) < 1$ can be assumed to hold for any $\epsilon > 0$, and we get that
$$(1 - F_X(\epsilon))^n \to 0,$$
as $n \to \infty$, and the assertion is proved.

b) We shall work using the basic definition of convergence in distribution. Thus we start by evaluating the distribution function $F_{nY_n}(x)$. Clearly $F_{nY_n}(x) = 0$ for $x \leq 0$. For an arbitrary $x > 0$
$$F_{nY_n}(x) = P(nY_n \leq x) = P\left(Y_n \leq \frac{x}{n}\right)$$
$$= P\left(\min(X_1, X_2, \ldots, X_n) \leq \frac{x}{n}\right) = 1 - P\left(\min(X_1, X_2, \ldots, X_n) > \frac{x}{n}\right).$$
By the same argument that was applied in case a) we obtain
$$= 1 - \left(1 - F_X\left(\frac{x}{n}\right)\right)^n.$$

Since the random variables have a density we have by definition of the first (right hand) derivative at 0, that
$$\frac{F_X\left(\frac{x}{n}\right) - F_X(0)}{\frac{x}{n} - 0} = c + \delta_n$$

since we have assumed the limit $\lim_{x \to 0} f_X(x) = c$, and where $\delta_n \to 0$ and as $n \to \infty$. As the random variables are positive (i.e. $F_X(0) = 0$) we get

$$F_X\left(\frac{x}{n}\right) = F_X\left(\frac{x}{n}\right) - F_X(0) = \frac{x}{n} \cdot c + \frac{x}{n} \cdot \delta_n,$$

Thus we have

$$F_{nY_n}(x) = 1 - \left(1 - \left(\frac{cx}{n} + \frac{x}{n} \cdot \delta_n\right)\right)^n$$

We set

$$c_n = c \cdot x + x\delta_n$$

and then, as $n \to \infty$,

$$c_n \to x \cdot c = c \cdot x.$$

Therefore

$$\left(1 - F_X\left(\frac{x}{n}\right)\right)^n = \left(1 - \left(\frac{x}{n} \cdot c + \frac{x}{n} \cdot \delta_n\right)\right)^n = \left(1 - \frac{c_n}{n}\right)^n.$$

Then a formula in section 13 of the Collection of Formulas entails that

$$\left(1 - \frac{c_n}{n}\right)^n \to e^{-c \cdot x}.$$

Thereby we have shown that

$$F_{nY_n}(x) = 1 - \left(1 - \left(\frac{x}{n} \cdot c + \frac{x}{n} \cdot \delta_n\right)\right)^n \to 1 - e^{-c \cdot x}.$$

Since $c > 0$ we recognize the function $F_X(x) = 1 - e^{-c \cdot x}, x > 0$ and $F_X(x) = 0, x \leq 0$ as the distribution function of $X \in \text{Exp}\left(\frac{1}{c}\right)$. Since $F_{nY_n}(x) = 0$ for $x \leq 0$ and since $F_X(x)$ is a continuous function everywhere (including 0), we have shown that

$$F_{nY_n}(x) \to F_X(x)$$

at every point of continuity of $F_X(x)$. By definition of convergence in distribution we have established that

$$nY_n \xrightarrow{d} \text{Exp}\left(\frac{1}{c}\right),$$

as $n \to \infty$, as was desired.

Uppgift 4

a) We apply the standard trick of re-writing the sought probability as

$$P\left(X(t) \leq X(t-1) + 4.0\right) = P\left(X(t) - X(t-1) \leq 4.0\right).$$

Let us introduce the auxiliary notation

$$Z = X(t) - X(t-1).$$

Since the underlying stochastic process $\mathbf{X} = \{X(t) \mid -\infty < t < \infty\}$ is Gaussian, $(X(t), X(t-1))$ is a bivariate Gaussian random variable and thus Z as defined above is a Gaussian random variable. Hence, in order to evaluate

$$P\left(X(t) - X(t-1) \leq 4.0\right) = P\left(Z \leq 4.0\right)$$

we need merely to find the mean and variance of Z. As far as the mean is concerned we have $E[Z] = 0$, since the means of $X(t)$ are $X(t-1)$ are the same $(= 0)$. For variance we apply one of the statements in the Collection of Formulas to obtain

$$\text{Var}(Z) = \text{Var}(X(t)) + \text{Var}(X(t)) - 2\text{Cov}(X(t), X(t-1)).$$

As the underlying process is weakly stationary and the means are zero, we get

$$\text{Var}(X(t)) = \text{Var}(X(t)) = R_{\mathbf{X}}(0) = \frac{1}{2}.$$

As the underlying process is weakly stationary and the means are zero, we get

$$\text{Cov}(X(t), X(t-1)) = R_{\mathbf{X}}(1) = \frac{1}{2}e^{-1}.$$

When we insert all of these in the right places we get

$$\text{Var}(Z) = 1 - e^{-1}.$$

Then we have

$$P(Z \leq 4.0) = P\left(\frac{Z}{\sqrt{1-e^{-1}}} \leq \frac{4.0}{\sqrt{1-e^{-1}}}\right).$$

Since $\frac{Z}{\sqrt{1-e^{-1}}} \in N(0,1)$, we have that

$$P(Z \leq 4.0) = \Phi\left(\frac{4.0}{\sqrt{1-e^{-1}}}\right),$$

where $\Phi(x)$ is the cumulative distribution function of the standard normal distribution. By some numerical work we can get

$$\Phi\left(\frac{4.0}{\sqrt{1-e^{-1}}}\right) = \Phi(0.5031) = 0.69.$$

$$\text{ANSWER a): } \underline{P(X(t) \leq X(t-1) + 4.0) = \Phi\left(\tfrac{4.0}{\sqrt{1-e^{-1}}}\right).}$$

b) Since a weakly stationary Gaussian process is also (strictly) stationary, we have $(X(t), X(t-1)) \overset{d}{=} (X(t+1), X(t))$, it follows that $P(X(t+1) \leq X(t) + 4.0) = P(X(t) \leq X(t-1) + 4.0)$.

$$\text{ANSWER b): } \underline{P(X(t+1) \leq X(t) + 4.0) = \Phi\left(\tfrac{4.0}{\sqrt{1-e^{-1}}}\right).}$$

Uppgift 5

The characteristic function is

$$\varphi_{W(T)}(s) = E\left[e^{isW(T)}\right].$$

It is known (Collection of Formulas about Poisson process) that $T \in \text{Exp}(\lambda)$. Then we use double expectation

$$E\left[e^{isW(T)}\right] = E\left[E\left[e^{isW(T)} \mid T\right]\right] = \int_0^\infty E\left[e^{isW(t)} \mid T = t\right] \lambda e^{-\lambda t} dt,$$

where we inserted the density of T. Since the Poisson process is independent of $\mathbf{W}$, T is independent of $\mathbf{W}$. This gives

$$E\left[e^{isW(T)}\right] = \int_0^\infty E\left[e^{isW(t)}\right] \lambda e^{-\lambda t} dt.$$

Because $W(t) \in N(0,t)$ we have

$$E\left[e^{isW(t)}\right] = e^{-s^2 t/2}.$$

When we substitute this expression in the integral above we get

$$E\left[e^{isW(T)}\right] = \int_0^\infty e^{-s^2 t/2} \lambda e^{-\lambda t} dt = \lambda \int_0^\infty e^{-\left(s^2/2+\lambda\right)t} dt$$

$$= \frac{2\lambda}{2\lambda + s^2} = \frac{1}{1 + \frac{s^2}{2\lambda}},$$

as was claimed.

b) The distribution of $W(T)$ is found by consulting the table of characteristic functions due to Allan Gut. There we find that

$$\frac{1}{1 + \frac{s^2}{2\lambda}}$$

is the characterstic function for $\text{La}(\frac{1}{\sqrt{2\lambda}})$.

ANSWER b): $\underline{W(T) \in \text{La}(\frac{1}{\sqrt{2\lambda}})}$.

Uppgift 6

a)

$$\sum_{i=1}^n \left(X_i - \overline{X}_n\right)^2 = \sum_{i=1}^n \left((X_i - \mu) - \left(\overline{X}_n - \mu\right)\right)^2$$

$$= \sum_{i=1}^n (X_i - \mu)^2 + \sum_{i=1}^n \left(\overline{X}_n - \mu\right)^2 - 2\left(\overline{X}_n - \mu\right) \sum_{i=1}^n (X_i - \mu).$$

We have that

$$\sum_{i=1}^n \left(\overline{X}_n - \mu\right)^2 = n\left(\overline{X}_n - \mu\right)^2,$$

and

$$\sum_{i=1}^n (X_i - \mu) = \sum_{i=1}^n X_i - n\mu = n\overline{X}_n - n\mu = n\left(\overline{X}_n - \mu\right).$$

Thus in the expression above

$$\sum_{i=1}^{n}(X_i-\mu)^2+\sum_{i=1}^{n}\left(\overline{X}_n-\mu\right)^2-2\left(\overline{X}_n-\mu\right)\sum_{i=1}^{n}(X_i-\mu)=$$

$$=\sum_{i=1}^{n}(X_i-\mu)^2+n\left(\overline{X}_n-\mu\right)^2-2n\left(\overline{X}_n-\mu\right)^2$$

$$=\sum_{i=1}^{n}(X_i-\mu)^2-n\left(\overline{X}_n-\mu\right)^2$$

as claimed.

b) From (1) and part a)

$$S_n^2=\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i-\overline{X}_n\right)^2=$$

$$=\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\mu)^2-\frac{n}{n-1}\left(\overline{X}_n-\mu\right)^2$$

We consider the first term in the right hand side.

$$\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\mu)^2=\frac{n}{n-1}\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2.$$

Since $X_1, X_2, \ldots$ are independent and identically distributed random variables with $E[X_i]=\mu$ and $\text{Var}[X_i]=\sigma^2<+\infty$, we get by the law of large numbers that

$$\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2\xrightarrow{P}E[X-\mu]^2=\text{Var}[X]=\sigma^2.$$

Since $\frac{n}{n-1}\to 1$, as $n\to\infty$,

$$\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\mu)^2=\frac{n}{n-1}\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2\xrightarrow{P}\sigma^2,$$

as $n\to+\infty$.
Now we show that the second term $\frac{n}{n-1}\left(\overline{X}_n-\mu\right)^2$ converges in probability to zero, and taken together with the result on the first term we have proved the claim in part b), when we invoke in addition theorem 6.2 of chapter 6 in Gut.
Since $X_1, X_2, \ldots$ are independent and identically distributed random variables with $E[X_i]=\mu$, we get by the law of large numbers

$$\overline{X}_n-\mu\xrightarrow{P}\mu-\mu=0.$$

Since $g(x)=x^2$ is a continuous function, we get that

$$\frac{n}{n-1}\left(\overline{X}_n-\mu\right)^2=\frac{n}{n-1}g\left(\overline{X}_n-\mu\right)\xrightarrow{P}g(0)=0,$$

as $n \to +\infty$. Thus we have established that

$$\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}_n\right)^2 \xrightarrow{P} \sigma^2$$

which is a consistency property of this standard estimator of variance.

c) The ratio

$$\frac{\sqrt{n}\left(\overline{X}_n - \mu\right)}{S_n},$$

plays an important role e.g. in construction of confidence intervals for μ. If $X_1, X_2, \ldots$ are independent and identically distributed Gaussian random variables, this ratio above has a Student distribution with $n-1$ degrees of freedom. We plan to study the pertinent asymptotics by means of the Cramér-Slutzky device.
First we write in the numerator

$$\sqrt{n}\left(\overline{X}_n - \mu\right) = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right)$$

$$= \sqrt{n}\frac{1}{n}\left(\sum_{i=1}^{n} X_i - n\mu\right) = \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n} X_i - n\mu\right)$$

and the **central limit theorem** gives

$$\frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n} X_i - n\mu\right) \xrightarrow{d} N(0,\sigma^2),$$

as $n \to +\infty$.
In the denominator we have for the continuous function $h(x) = \sqrt{x}$ from b) that

$$S_n = h\left(S_n^2\right) \xrightarrow{P} h(\sigma^2) = \sigma,$$

as $n \to +\infty$. Thus the conditions required in the **Cramér-Slutzky** theorem hold and the conlusion is thereby that

$$\frac{\sqrt{n}\left(\overline{X}_n - \mu\right)}{S_n} \xrightarrow{d} \frac{1}{\sigma} Z,$$

where $Z \in N(0,\sigma^2)$. Therefore

$$\frac{\sqrt{n}\left(\overline{X}_n - \mu\right)}{S_n} \xrightarrow{d} N(0,1),$$

as $n \to +\infty$.

ANSWER: The limiting distribution is $N(0,1)$.