KTH Matematik

Avd. Matematisk statistik

TENTAMEN I SF2940 SANNOLIKHETSTEORI/EXAM IN SF2940 PROBABILITY THEORY, Tuesday December 18, 2018, 08.00-13.00.

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Tillåtna hjälpmedel/Permitted means of assistance: Appendix 2 in A. Gut: An Intermediate Course in Probability, Formulas for probability theory SF2940, L. Råde & B. Westergren: Mathematics Handbook for Science and Engineering and pocket calculator.

All used notation must be explained and defined. Reasoning and the calculations must be so detailed that they are easy to follow. Each problem yields max 10 p. You may apply results stated in a part of an exam question to another part of the exam question even if you have not solved the first part. A preliminarily lower bound of 25 points will guarantee a passing result.
Solutions to the exam questions will be available at http://www.math.kth.se/matstat/gru/sf2940/ starting from Monday January 14, 2019.

Good luck!

Problem 1

Let $X_1, X_2, X_3, \ldots$ be independent, identically distributed random variables with $E[X_1] = -2$ and $\mathrm{Var}[X_1] = 4$. Find the limiting distribution of

$$\sqrt{n}\frac{X_1 + X_2 + \ldots + X_n + 2n}{X_1^2 + X_2^2 + \ldots + X_n^2} \quad \text{as } n \to +\infty.$$

Each step should be carefully motivated. (10 p)

Problem 2

Suppose X and Y are square-integrable random variables such that $E[X|Y] = Y$ and $E[Y|X] = X$. Show that $X = Y$ almost surely. (10 p)

Problem 3

The random variables $X_1, X_2, \ldots$ are independent and identically distributed non-negative and take values in $\{0, 1, 2, 3, \ldots\}$. The random variable $N \in \mathrm{Po}(\lambda)$ is independent of $X_1, X_2, \ldots$ Set

$$S_N = X_1 + X_2 + \ldots + X_N, \quad S_0 = 0.$$

If $S_N \in \mathrm{Po}(\frac{\lambda}{3})$, what is the distribution of an X_k? (10 p)

Problem 4

Let $(X_n,\, n \geq 1)$ be a sequence of random variables and set $\mu_n = E[X_n]$ and $\sigma_n^2 = \mathrm{Var}(X_n)$, for every $n \geq 1$.

(a) Assume that $\lim_{n\to\infty} \mu_n = \mu$ and $\lim_{n\to\infty} \sigma_n = 0$.

Show that $X_n \xrightarrow{P} \mu$ as $n \to +\infty$.

(b) Assume that each X_n is log-normally distributed with parameters $\left(\left(1-\frac{2}{n}\right)^n, \frac{1}{n}\right)$. Determine the constant a such that

$$X_n \xrightarrow{P} a \quad \text{as } n \to +\infty.$$

Hint: If $Y \geq 0$ a.s. then for every $\epsilon > 0$, $P(Y \geq \epsilon) \leq \frac{1}{\epsilon^2} E[Y^2]$ (Markov inequality). (10 p)

Problem 5

Let $X \in \mathrm{Po}(\lambda)$, $\lambda > 0$.

(a) Compute $L(t) := E[e^{itX} \big| X > 0]$, $\quad t \in \mathbb{R}$.

(b) Compute $E[X^2 \big| X > 0]$.

(10 p)

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Suggested solutions to the exam of Tuesday December 18, 2018. The problems can be solved using other methods than the suggested below.

Problem 1

We have

$$S_n := \sqrt{n}\frac{\sum_{i=1}^n X_i + 2n}{\sum_{i=1}^n X_i^2} = \frac{\frac{\sqrt{n}}{n}\left(\sum_{i=1}^n X_i + 2n\right)}{\sum_{i=1}^n \frac{X_i^2}{n}}.$$

In view of the Law of Large Numbers, we have

$$\sum_{i=1}^n \frac{X_i^2}{n} \xrightarrow{P} E[X_1^2] = \mathrm{Var}(X_1) + (E[X_1])^2 = 8.$$

Moreover, the Central Limit Theorem yields

$$\frac{\sqrt{n}}{n}\left(\sum_{i=1}^n X_i + 2n\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^n (X_i - E[X_i]) \xrightarrow{d} N(0,4).$$

By Cramèr-Slutsky's theorem, we finally obtain

$$S_n \xrightarrow{d} \frac{1}{8}N(0,4) = N(0,\frac{1}{16}).$$

Problem 2

It suffices to show that $E[(X-Y)^2] = 0$, since then $X = Y$ a.s.
We have

$$E[(X-Y)^2] = E[X^2] + E[Y^2] - 2E[XY].$$

Note that since $E[Y|X] = X$, we may write $E[X^2] = E[XX] = E[XE[Y|X]] = E[XY]$. We also have $E[Y^2] = E[YY] = E[YE[X|Y]] = E[XY]$, since $E[X|Y] = Y$.
Therefore,

$$E[(X-Y)^2] = E[XY] + E[XY] - 2E[XY] = 0.$$

Problem 3

We have

$$\varphi_{S_N}(t) = E[e^{itS_N}] = g_N(\varphi_{X_1}(t)).$$

Since $g_N(s) = e^{\lambda(s-1)}$ and, by assumption, $\varphi_{S_N}(t) = e^{\frac{\lambda}{3}(e^{it}-1)}$, we have

$$\exp\left(\frac{\lambda}{3}(e^{it}-1)\right) = \exp\left(\lambda(\varphi_{X_1}(t)-1)\right).$$

That is

$$\frac{\lambda}{3}(e^{it}-1)=\lambda(\varphi_{X_1}(t)-1),$$

or

$$\varphi_{X_1}(t)=\frac{2}{3}+\frac{e^{it}}{3}=\varphi_{\mathrm{Be}(\frac{1}{3})}(t).$$

Hence, $X_k \in \mathrm{Be}(\frac{1}{3})$.

Problem 4

a) Using the hint with $Y=|X_n-\mu|$, it suffices to show that $E[(X_n-\mu)^2] \longrightarrow 0$ as $n\to+\infty$. We have

$$E[(X_n-\mu)^2]=E[X_n^2]-2\mu_n\mu+\mu^2=\sigma_n^2+\mu_n^2-2\mu_n\mu+\mu^2 \longrightarrow \mu^2-2\mu^2+\mu^2=0 \quad \text{as } n\to+\infty.$$

Use the hint (Markov inequality) to obtain, for every $\epsilon>0$,

$$\lim_{n\to+\infty} P(|X_n-\mu|>\epsilon)\le \frac{1}{\epsilon}\lim_{n\to+\infty} E[(X_n-\mu)^2]=0,$$

i.e. $X_n \xrightarrow{P} \mu$ as $n\to+\infty$.

b) Since $X_n \in LN\left(\left(1-\frac{2}{n}\right)^n, \frac{1}{n}\right)$, then

$$\mu_n=e^{\left(1-\frac{2}{n}\right)^n+\frac{1}{2n}} \longrightarrow e^{e^{-2}}, \quad \sigma_n=e^{2\left(1-\frac{2}{n}\right)^n}\left(e^{\frac{2}{n}}-e^{\frac{1}{n}}\right)\longrightarrow 0 \quad \text{as } n\to+\infty.$$

Therefore, $X_n \xrightarrow{P} e^{e^{-2}}$ as $n\to+\infty$.

Problem 5

$X\in \mathrm{Po}(\lambda)$ i.e. $P(X=n)=e^{-\lambda}\frac{\lambda^n}{n!}$, , $n=0,1,2,\ldots$. Therefore, $P(X>0)=1-P(X=0)=1-e^{-\lambda}$.

a)

$$\begin{aligned} L(t) &= E[e^{itX}\big|X>0]=\tfrac{1}{P(X>0)}E[e^{itX}I_{\{X>0\}}]=\tfrac{1}{P(X>0)}\sum_{n\ge 0}E[e^{itX}I_{\{X>0\}}|X=n]P(X=n)\\ &= \tfrac{1}{P(X>0)}\sum_{n\ge 1}E[e^{itX}|X=n]P(X=n)=\tfrac{1}{P(X>0)}\sum_{n\ge 1}e^{itn}e^{-\lambda}\tfrac{\lambda^n}{n!}=\tfrac{e^{-\lambda}}{P(X>0)}\sum_{n\ge 1}\tfrac{\left(\lambda e^{it}\right)^n}{n!}\\ &= \tfrac{e^{-\lambda}}{P(X>0)}\left(\sum_{n\ge 1}\tfrac{\left(\lambda e^{it}\right)^n}{n!}-1\right)=\tfrac{e^{-\lambda}}{P(X>0)}\left(e^{\lambda e^{it}}-1\right)\\ &= \tfrac{e^{-\lambda}}{1-e^{-\lambda}}\left(e^{\lambda e^{it}}-1\right). \end{aligned}$$

b) Note that $E[X^2\big|X>0]=-L''(0)$.
We have

$$-L''(t)=\frac{\lambda e^{-\lambda}}{1-e^{-\lambda}}(\lambda e^{it}+1)e^{it}e^{\lambda e^{it}}.$$

Therefore,

$$E[X^2\big|X>0]=-L''(0)=\frac{\lambda+\lambda^2}{1-e^{-\lambda}}.$$