Ex 11 section 3.8.3 p 108 (**)

Let $X, Y \in \mathrm{Exp}(1)$ be independent. It means that their joint distribution is given by

$$f_{X,Y}(x,y) = e^{-x-y} \mathbb{1}_{x>0} \mathbb{1}_{y>0}$$

Show that $\frac{X}{X+Y} \in U(0,1)$?

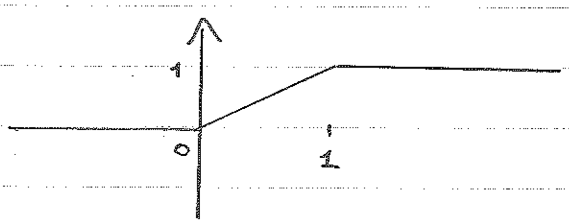
$$\mathbb{P}\left[\frac{X}{X+Y} \leq t\right] = \int_0^\infty \int_0^\infty \mathbb{1}_{x \leq t(x+y)} \, e^{-x-y} \, dx \, dy$$

Make the change of variable $u = x+y$ $(du = dy)$; by Fubini's theorem, we get:

$$\text{"} \quad = \int_0^\infty e^{-u} du \underbrace{\int_0^u \mathbb{1}_{x \leq tu} \, dx}_{\substack{= 0 \text{ if } t \leq 0 \\ = u \min\{t;1\} \text{ if } t>0}}$$

$$\text{"} \quad = \mathbb{1}_{t>0} \min\{t;1\} \underbrace{\int_0^\infty u e^{-u} du}_{=1}$$

$$\mathbb{P}\left[\frac{X}{X+Y} \leq t\right] = \mathbb{1}_{t>0} \min\{t;1\}$$



It shows that the random variable $\frac{X}{X+Y} \in U(0,1)$ $\square$