Ex 5 section 5.8.1 p 157

Let $X_1, X_2, \ldots$ be iid with distribution F_X

Let N be an $\mathbb{N}$-valued random variable independent of the sequence $(X_n)_{n=1}^{\infty}$.

Set $\quad Y = \max\{X_1, X_2, \ldots, X_N\}$

Show that $\quad F_Y(y) = g_N(F_X(y))$?

By the Law of total probability

$$\mathbb{P}[Y \leq y] = \sum_{n=1}^{\infty} \mathbb{P}[Y \leq y \mid N = n] \; \mathbb{P}[N = n]$$

We have

$$\mathbb{P}[Y \leq y \mid N = n] = \mathbb{P}[\max\{X_1, \ldots, X_n\} \leq y]$$

$$N \perp\!\!\!\perp \sigma(X_1, \ldots, X_n) \rightsquigarrow \quad = \mathbb{P}[X_1 \leq y \,;\, X_2 \leq y \,;\, \ldots \,;\, X_n \leq y]$$

$$= F_X(y)^n$$

$X_1, \ldots, X_n$ are iid

Then

$$\mathbb{P}[Y \leq y] = \sum_{n=1}^{\infty} F_X(y)^n \; \mathbb{P}[N = n]$$

$$= g_N(F_X(y)) \qquad \square$$