Ex 15 sect 6.8.1 p 183 (*)

$X_1, X_2, \dots$ be i.i.d. $L(a)$ $\longrightarrow$ $\begin{cases} f_{X_1}(x) = \frac{1}{2a} e^{-|x|/a} \\ \varphi_{X_1}(t) = \frac{1}{1+(at)^2} \end{cases}$

N be an ind. $\mathrm{Poi}(m)$

Set $S_N = X_1 + \dots + X_N$.

Let $m \to \infty$ and $a \to 0$ in the regime $ma^2 \to 1$, we want to prove that $S_N \xrightarrow{d} N(0,2)$

$$\varphi_{S_N}(t) = g_N(\varphi_{X_1}(t))$$

$$= \exp\left(m\left(\frac{1}{1+(at)^2} - 1\right)\right) = \exp\left(-\frac{ma^2 t^2}{1+a^2t^2}\right)$$

$$\lim_{\substack{m\to\infty \\ a=\sqrt{\frac{1}{m}}}} \varphi_{S_N}(t) = e^{-t^2} = e^{-\sigma^2 t^2/2} \quad \text{where } \sigma^2 = 2$$

$$\Longrightarrow \boxed{S_N \xrightarrow{d} N(0,2)}$$

Ex 16 p183 (*)

$$g_{S_N}(t) = g_N(g_{X_1}(t)) = \exp\left(\lambda\left(e^{\mu(t-1)} - 1\right)\right)$$

$$\lambda\left(e^{\mu(t-1)} - 1\right) = \lambda\mu(t-1) + O(\lambda\mu^2)$$

$$\longrightarrow c(t-1) + O(\mu) \quad \text{as} \quad \begin{matrix}\mu \to 0 \\ \lambda\mu \to c > 0\end{matrix}$$

$$\lim_{\substack{\mu\to 0 \\ \lambda = c/\mu}} g_{S_N}(t) = \exp(c(t-1)) \quad \forall t \in \mathbb{R} \quad \Longrightarrow \boxed{S_N \xrightarrow{d} \mathrm{Poi}(c)}$$