Avd. Matematisk statistik

KTH Teknikvetenskap

A Limit Formula in Calculus Applied to Probability Theory SF2940

Timo Koski
Institutionen för matematik
Kungliga tekniska högskolan (KTH) , Stockholm

15 september 2011

1 The statement

The following statement (COLLECTION OF FORMULAS FOR SF2940 section 13.1) (proposition 1.1 below) appears under various guises in several of the proofs and the exercises of A. Gut: An Intermediate Course in Probability, 2nd Ed.. We refer e.g. to the proofs of Theorem 5.1 (p. 162) and the proof of Theorem 5.2 (pp. 162-163). We shall now, for the sake of definiteness, prove the statement.

Proposition 1.1

$$c_n \to c \Rightarrow \left(1 + \frac{c_n}{n}\right)^n \to e^c, \qquad \text{as } n \to \infty. \tag{1.1}$$

Proof Let us consider

$$n \ln\left(1 + \frac{c_n}{n}\right) = c_n \cdot \frac{\ln\left(1 + \frac{c_n}{n}\right)}{\frac{c_n}{n}}. \tag{1.2}$$

Then we recall a standard limit in calculus (or, the derivative of $\ln x$ at $x = 1$)

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1.$$

Since $c_n \to c$ by assumption, we have that $\frac{c_n}{n} \to 0$, as $n \to \infty$. Thus

$$\lim_{n \to \infty} \frac{\ln\left(1 + \frac{c_n}{n}\right)}{\frac{c_n}{n}} = 1,$$

we get in (1.2) that

$$\lim_{n \to \infty} n \ln\left(1 + \frac{c_n}{n}\right) = c.$$

Let $x_n = n \ln\left(1 + \frac{c_n}{n}\right)$. Since e^x is a continuous function we get

$$\left(1 + \frac{c_n}{n}\right)^n = e^{x_n} \to e^c.$$

■

The proof above is strictly speaking valid for sequences of real numbers, but an analogous proof holds for complex numbers.

There are other arguments that often are supplanting the proofs of Theorem 5.1 (p. 162) and the proof of Theorem 5.2 (pp. 162-163) in Gut.

2 Some Inequalities

Lemma 2.1 For any complex numbers w_i, z_i, if $|w_i| \leq 1$ and $|z_i| \leq 1$, then

$$|\prod_{i=1}^{n} z_i - \prod_{i=1}^{n} w_i| \leq \sum_{i=1}^{n} |z_i - w_i|. \tag{2.3}$$

Proof We have the identity

$$| \prod_{i=1}^{n} z_i - \prod_{i=1}^{n} w_i |=| (z_n - w_n) \prod_{i=1}^{n-1} z_i + w_n \left(\prod_{i=1}^{n-1} z_i - \prod_{i=1}^{n-1} w_i \right) |$$

and the right hand side of this inequality is upper bounded by

$$\leq| z_n - w_n | + | \prod_{i=1}^{n-1} z_i - \prod_{i=1}^{n-1} w_i |,$$

since $|w_n| \leq 1$ and $|z_i| \leq 1$. Then we use an induction hypothesis. ■

Lemma 2.2 For any complex numbers u and v,

$$|u^n - v^n| \leq |u - v| n \max(|u|, |v|)^{n-1} \tag{2.4}$$

Proof We have the identity

$$u^n - v^n = (u - v)u^{n-1} + v(u^{n-1} - v^{n-1})$$

and then

$$|u^n - v^n| \leq |u - v||u^{n-1}| + |v||u^{n-1} - v^{n-1}|. \tag{2.5}$$

Our induction hypothesis is that

$$|u^{n-1} - v^{n-1}| \leq |u - v|(n - 1) \max(|u|, |v|)^{n-2}.$$

When we apply this in right hand side of (2.5) we get

$$|u^n - v^n| \leq |u - v||u^{n-1}| + |v||u - v|(n - 1) \max(|u|, |v|)^{n-2}.$$

We note that

$$|u^{n-1}| \leq \max(|u|, |v|)^{n-1}$$

and that

$$|v| \cdot \max(|u|, |v|)^{n-2} \leq \max(|u|, |v|) \cdot \max(|u|, |v|)^{n-2} = \max(|u|, |v|)^{n-1}.$$

Thus we have obtained

$$|u^n - v^n| \leq |u - v| \max(|u|, |v|)^{n-1} + |u - v|(n - 1) \max(|u|, |v|)^{n-1},$$

which proves (2.4) as asserted. ■

3 Applications

In Gut loc.cit. the situation corresponding to (1.1) is often encountered as

$$\left(1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right)^n \to e^{-t^2}, \tag{3.6}$$

where $o(t)$ is a function such that $\lim_{t\to 0} \frac{o(t)}{t}$ (Landau's o -notation).

1. Let us set
$$c_n = -\left(\frac{t^2}{2} + n \cdot o\left(\frac{t^2}{n}\right)\right).$$
Then $n \cdot o\left(\frac{t}{n}\right) = \frac{o\left(\frac{t}{n}\right)}{\frac{1}{n}} \to 0$, as $n \to \infty$. Thus $c_n \to -\frac{t^2}{2}$, as $n \to \infty$, and (3.6) follows by (1.1), since
$$\frac{c_n}{n} = -\left(\frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right).$$

2. Let us now check (3.6) using the inequalities in the preceding section. With regard to lemma 2.1 we take for all i
$$z_i = \left(1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right)$$
and
$$w_i = \left(1 - \frac{t^2}{2n}\right).$$
Then $|w_i| \leq 1$ and $|z_i| \leq 1$ and
$$|\prod_{i=1}^{n} z_i - \prod_{i=1}^{n} w_i| = |\left(1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right)^n - \left(1 - \frac{t^2}{2n}\right)^n|,$$
and in view of the lemma 2.1 above we get
$$|\left(1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right)^n - \left(1 - \frac{t^2}{2n}\right)^n| \leq n \mid o\left(\frac{t^2}{n}\right) \mid =$$
$$= \mid \frac{o\left(\frac{t^2}{n}\right)}{\frac{1}{n}} \mid .$$
When $n \to \infty$, $\frac{o\left(\frac{t^2}{n}\right)}{\frac{1}{n}} \to 0$, by definition of Landau's o. Since
$$\left(1 - \frac{t^2}{2n}\right)^n \to e^{-t^2/2}, \qquad \text{as } n \to \infty,$$
it now follows that as $n \to \infty$
$$\left(1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right)^n \to e^{-t^2/2},$$
as was to be proved.

3. If $| u |< 1$ and $| v |< 1$, then we get in (2.4)

$$|u^n - v^n| \leq |u - v|n$$

and thus with

$$u = \left(1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right), v = \left(1 - \frac{t^2}{n}\right),$$

we obtain again as $n \to \infty$, that

$$\left(1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right)^n \to e^{-t^2/2}.$$