EXAMINATION IN SF2942 PORTFOLIO THEORY AND RISK MANAGEMENT

*Date:* 2014-10-21, 8:00-13:00

Suggested solutions

- - - - - - - - - - - - - - - - - - - -

## Problem 1

(a) Let $d_k$, $k = 1, 2, 3$, be the discount factors. We know that the price $P$ at $t = 0$ of a bond with coupons $c_k$, face value $F$ and maturity time $n$ is given by

$$P = \sum_{k=1}^{n} c_k d_k + F d_n.$$

If we know $d_k$, we can get the zero rates from the relation

$$d_k = e^{-r_k k} \quad \Leftrightarrow \quad r_k = -\frac{1}{k} \ln d_k.$$

Using the given bond information we can calculate $d_1$, $d_2$ and $d_3$. Hence, the zero rates we can calculate are $r_1$, $r_2$ and $r_3$. We have

$$\begin{cases} 5d_1 + 5d_2 + 105d_3 &= 104 \\ 100d_1 &= 98 \\ 2d_1 + 102d_2 &= 100, \end{cases}$$

and the solution to this system of equations is

$$d_1 = 0.980, \ d_2 = 0.961 \text{ and } d_3 = 0.898.$$

Using this we get the zero rates

$$r_1 = 2.02\% \ r_2 = 1.98\% \text{ and } r_3 = 3.58\%.$$

(b) The arbitrage free price $P$ of a stream of cash flows $c_k$ is in general given by

$$P = \sum_{k=1}^{n} c_k d_k.$$

Using the $d_k$'s from (a) we get the following price of our given cash flow:

$$P = 50 \cdot 0.980 + 20 \cdot 0.961 + 75 \cdot 0.898 = 135.5765 \approx 135.6.$$

(c) For a stream of cash flows $(c_1, \ldots, c_n)$ with price $P$, the IRR $r_0$ is the rate fulfilling

$$P = \sum_{k=1}^{n} c_k e^{-r_0 \cdot k}.$$

Let $d = e^{-r_0}$. Then the previous equation can be written

$$P = \sum_{k=1}^{n} c_k d^k.$$

In our case we get

$$135.6 = 50d + 20d^2 + 75d^3.$$

The solution to this eqution is

$$d = 0.9694,$$

and we get

$$r_0 = -\ln d = 3.11\%.$$

**Problem 2**

(a) Set $L = S_T^{\beta}$. We want to find the payoff $A$ in the set

$$\left\{ A \,\middle|\, A = h_0 + hS_T,\ (h_0, h) \in \mathbb{R}^2 \right\}$$

that minimizes $E\left[(L-A)^2\right]$. We know that the optimal $(h_0, h)$ is given by

$$h = \frac{\mathrm{Cov}(L, S_T)}{\mathrm{Var}(S_T)} \text{ and } h_0 = E\left[L\right] - hE\left[S_T\right].$$

We can write

$$\ln(S_T/S_0) = \mu T + \sigma\sqrt{T}Z,$$

where $Z \sim N(0,1)$, or

$$S_T = S_0 e^{\mu T + \sigma\sqrt{T}Z}.$$

When $S_T$ is lognormally distributed as here, we have

$$E\left[S_T^a\right] = E\left[\left(S_0 e^{\mu T + \sigma\sqrt{T}Z}\right)^a\right] = S_0^a E\left[e^{a\mu T + a\sigma\sqrt{T}Z}\right] = S_0^a e^{a\mu T + a^2\sigma^2 T/2}. \ (\star)$$

Now

$$\begin{aligned}
\mathrm{Cov}(L, S_T) &= \mathrm{Cov}(S_T^{\beta}, S_T) \\
&= E\left[S_T^{\beta} S_T\right] - E\left[S_T^{\beta}\right] E\left[S_T\right] \\
&= E\left[S_T^{\beta+1}\right] - E\left[S_T^{\beta}\right] E\left[S_T\right] \\
&= \{\text{Use } (\star)\} \\
&= S_0^{\beta+1} e^{(\beta+1)\mu T + (\beta+1)^2\sigma^2 T/2} - S_0^{\beta} e^{\beta\mu T + \beta^2\sigma^2 T/2} \cdot S_0 e^{\mu T + \sigma^2 T/2} \\
&= S_0^{\beta+1} e^{(\beta+1)\mu T + (\beta^2+1)\sigma^2 T/2} \left[e^{\beta\sigma^2 T} - 1\right],
\end{aligned}$$

$$
\begin{aligned}
\mathrm{Var}(S_T) &= E\left[S_T^2\right] - E\left[S_T\right]^2 \\
&= \{\text{Use } (\star)\} \\
&= S_0^2 e^{2\mu T + 2\sigma^2 T} - \left(S_0 e^{\mu T + \sigma^2 T/2}\right)^2 \\
&= S_0^2 e^{2\mu T + 2\sigma^2 T} - S_0^2 e^{2\mu T + \sigma^2 T} \\
&= S_0^2 e^{2\mu T + \sigma^2 T}\left[e^{\sigma^2 T} - 1\right]
\end{aligned}
$$

and finally, using $(\star)$ again,

$$
E\left[S_T^\beta\right] = S_0^\beta e^{\beta\mu T + \beta^2\sigma^2 T/2}.
$$

We now get

$$
\begin{aligned}
h &= \frac{\mathrm{Cov}(S_T^\beta, S_T)}{\mathrm{Var}(S_T)} \\
&= S_0^{\beta-1} e^{(\beta-1)\mu T + (\beta^2-1)\sigma^2 T/2} \frac{e^{\beta\sigma^2 T} - 1}{e^{\sigma^2 T} - 1}
\end{aligned}
$$

and

$$
\begin{aligned}
h_0 &= E\left[S_T^\beta\right] - hE\left[S_T\right] \\
&= S_0^\beta e^{\beta\mu T + \beta^2\sigma^2 T/2} - S_0^{\beta-1} e^{(\beta-1)\mu T + (\beta^2-1)\sigma^2 T/2} \frac{e^{\beta\sigma^2 T} - 1}{e^{\sigma^2 T} - 1} S_0 e^{\mu T + \sigma^2 T/2} \\
&= S_0^\beta e^{\beta\mu T + \beta^2\sigma^2 T/2}\left[1 - \frac{e^{\beta\sigma^2 T} - 1}{e^{\sigma^2 T} - 1}\right] \\
&= S_0^\beta e^{\beta\mu T + \beta^2\sigma^2 T/2} \cdot \frac{e^{\sigma^2 T} - e^{\beta\sigma^2 T}}{e^{\sigma^2 T} - 1}.
\end{aligned}
$$

(b) The hedging error is given by $\hat{A} - L$, where $\hat{A}$ is the payoff when we use the optimal portfolio $(h_0, h)$ from (a). We get

$$
\begin{aligned}
\mathrm{Var}(\hat{A} - L) &= \mathrm{Var}(\hat{A}) - 2\mathrm{Cov}(\hat{A}, L) + \mathrm{Var}(L) \\
&= h^2 \mathrm{Var}(S_T) - 2h\mathrm{Cov}(S_T, L) + \mathrm{Var}(L) \\
&= \frac{\mathrm{Cov}(S_T, L)^2}{\mathrm{Var}(S_T)^2}\mathrm{Var}(S_T) - 2\frac{\mathrm{Cov}(S_T, L)}{\mathrm{Var}(S_T)}\mathrm{Cov}(S_T, L) + \mathrm{Var}(L) \\
&= -\frac{\mathrm{Cov}(S_T, S_T^\beta)^2}{\mathrm{Var}(S_T)} + \mathrm{Var}(S_T^\beta)
\end{aligned}
$$

Using results from (a) we get

$$
-\frac{\mathrm{Cov}(S_T, S_T^\beta)^2}{\mathrm{Var}(S_T)} = -S_0^{2\beta} e^{2\beta\mu T + \beta^2\sigma^2 T} \frac{\left(e^{\beta\sigma^2 T} - 1\right)^2}{e^{\sigma^2 T} - 1}
$$

and

$$\mathrm{Var}(S_T^\beta) = S_0^{2\beta} e^{2\beta\mu T + \beta^2\sigma^2 T}\left(e^{\beta^2\sigma^2 T} - 1\right).$$

Finally we get

$$\mathrm{Var}(\hat{A} - L) = S_0^{2\beta} e^{2\beta\mu T + \beta^2\sigma^2 T}\left[e^{\beta^2\sigma^2 T} - 1 - \frac{\left(e^{\beta\sigma^2 T} - 1\right)^2}{e^{\sigma^2 T} - 1}\right]$$

(c) The certainty equivalent $C$ in general satisfies

$$u(C) = E\left[u(X)\right],$$

and here

$$X = S_T^\beta \quad \text{and} \quad u(x) = \ln x.$$

We get

$$\ln C = E\left[\ln\left(S_T^\beta\right)\right] = \beta E\left[\ln S_T\right] = \beta(\ln S_0 + \mu T),$$

and hence

$$C = S_0^\beta e^{\beta\mu T}.$$

**Problem 3**

(a) The coefficient of absolute risk aversion is given by

$$A(x) = -\frac{u''(x)}{u'(x)} = -\frac{-\frac{3}{4}x^{-5/2}}{\frac{1}{2}x^{-3/2}} = \frac{3}{2}\cdot\frac{1}{x}.$$

(b) The problem we want to solve is

$$\left[\begin{array}{ll} \max & E\left[u\left(\sum_{k=1}^n w_k\theta_k X_k\right)\right] \\ \text{s.t.} & \sum_{k=1}^n w_k = V_0, \end{array}\right.$$

where

$$u(x) = -\frac{1}{\sqrt{x}},\ n = 3,\ \theta_1 = 2.5,\ \theta_2 = 3.25,\ \theta_3 = 2.85,\ V_0 = 50$$

and $X_k = 1$ if horse $k$ wins, and 0 otherwise.

We can write the objective function as

$$\sum_{k=1}^n p_k u(w_k\theta_k),$$

where $p_k = 1/3,\ k = 1, 2, 3$. The Lagrangian is

$$L = \sum_{k=1}^n p_k u(w_k\theta_k) + \lambda\left(V_0 - \sum_{k=1}^n w_k\right),$$

and the first order conditions are

$$\begin{aligned}\frac{\partial L}{\partial w_k} &= p_k\theta_k u'(w_k\theta_k) - \lambda = 0, \ k = 1,2,3 \\ \frac{\partial L}{\partial \lambda} &= V_0 - \sum_{k=1}^{n} = 0.\end{aligned}$$

Using these conditions, we get

$$w_k = \frac{1}{\theta_k}(u')^{-1}\left(\lambda\frac{1}{p_k\theta_k}\right), \ k = 1,2,3,$$

and

$$V_0 = \sum_{k=1}^{n}\frac{1}{\theta_k}(u')^{-1}\left(\lambda\frac{1}{p_k\theta_k}\right).$$

Now

$$u'(x) = \frac{1}{2}\cdot\frac{1}{x^{3/2}} \ \text{ and } \ (u')^{-1}(x) = \left(\frac{1}{2x}\right)^{2/3}.$$

It follows that

$$w_k = \frac{1}{\theta_k}\left(\frac{p_k\theta_k}{2\lambda}\right)^{2/3} = \frac{p_k^{2/3}}{\theta_k^{1/3}}\cdot\left(\frac{1}{2\lambda}\right)^{2/3}, \ k = 1,2,3,$$

and

$$V_0 = \sum_{k=1}^{3} w_k = \left(\frac{1}{2\lambda}\right)^{2/3}\sum_{k=1}^{3}\frac{p_k^{2/3}}{\theta_k^{1/3}}.$$

Hence

$$w_k = V_0\cdot\frac{\frac{p_k^{2/3}}{\theta_k^{1/3}}}{\sum_{i=1}^{3}\frac{p_i^{2/3}}{\theta_i^{1/3}}},$$

and with the given parameter values we get

$$(w_1, w_2, w_3) = (17.40, 15.94, 16.66).$$

(c) The Arrow-Debreu prices are given by

$$q_1 = 1/\theta_1 = 0.400, \ q_2 = 1/\theta_2 = 0.308 \text{ and } q_3 = 1/\theta_3 = 0.351.$$

If we short sell $q_k$ of outcome $k$ for $k = 1,2,3$, then we get

$$q_1 + q_2 + q_3 = 1.059 > 1$$

at time 0. This money is put into the bank account, and hence the total cash flow today is zero. After the match we have 1.059 in the bank, and need to pay out 1 unit irrespectivley of the outcome of the match, leaving us with the cash flow 0.059 with certainty after the match. This is an arbitrage opportunity.

## Problem 4

Here we assume that all returns are net returns. You could assume that the returns are gross returns, as in the course book, and then you will get a different answer to some of the questions.

(a) We want to solve the problem

$$\left[\begin{array}{ll} \min & \frac{1}{2}w^T\Sigma w \\ \text{s.t.} & w^T\mathbf{1} = V_0, \end{array}\right.$$

where $V_0 = 100$. The Lagrangian is

$$L = \frac{1}{2}w^T\Sigma w + \lambda(V_0 - w^T\mathbf{1}),$$

and the first order conditions are

$$\begin{aligned} \nabla L &= \Sigma w - \lambda\mathbf{1} = 0 \\ \frac{\partial L}{\partial \lambda} &= V_0 - w^T\mathbf{1} = 0. \end{aligned}$$

from which we get

$$w = \lambda\Sigma^{-1}\mathbf{1}$$

and

$$V_0 = \lambda\mathbf{1}^T\Sigma^{-1}\mathbf{1}.$$

Hence

$$w = \frac{V_0}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}\Sigma^{-1}\mathbf{1}.$$

Now

$$\Sigma^{-1} = \begin{bmatrix} 16.67 & -2.778 \\ -2.778 & 11.57 \end{bmatrix}$$

and we get

$$w = \frac{100}{22.69}\begin{bmatrix} 13.89 \\ 8.796 \end{bmatrix} = \begin{bmatrix} 61.2 \\ 38.8 \end{bmatrix}.$$

(b) The mean and variance of the portfolio from (a) is given by

$$\mu_{\text{mvp}} = w^T\mu = 11.9$$

and

$$\sigma^2_{\text{mvp}} = w^T\Sigma w = 440.8$$

respectively.

(c) Now the problem is

$$\left[\begin{array}{ll} \min & \frac{1}{2} w^T \Sigma w \\ \text{s.t.} & w^T \mu = \mu_0 V_0 \\ & w^T \mathbf{1} = V_0, \end{array}\right.$$

**Remark.** The first condition can be replaced with

$$w^T(\mathbf{1} + \mu) = (1 + \mu_0)V_0$$

and (in view if the second condition) we will still get the same answer.

The Lagrangian is

$$L = \frac{1}{2} w^T \Sigma w + \lambda_1 \big(\mu_0 V_0 - w^T \mu\big) + \lambda_2 \big(V_0 - w^T \mathbf{1}\big),$$

and the first order conditions are

$$\begin{aligned} \nabla L &= \Sigma w - \lambda_1 \mu - \lambda_2 \mathbf{1} = 0 \\ \frac{\partial L}{\partial \lambda_1} &= \mu_0 V_0 - w^T \mu = 0 \\ \frac{\partial L}{\partial \lambda_2} &= V_0 - w^T \mathbf{1} = 0 \end{aligned}$$

The optimal portfolio is

$$w = \lambda_1 \Sigma^{-1} \mu + \lambda_2 \Sigma^{-1} \mathbf{1},$$

and the multipliers are determined by

$$\begin{bmatrix} \mu^T \Sigma^{-1} \mu & \mu^T \Sigma^{-1} \mathbf{1} \\ \mathbf{1}^T \Sigma^{-1} \mu & \mathbf{1}^T \Sigma^{-1} \mathbf{1} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \mu_0 V_0 \\ V_0 \end{bmatrix}.$$

Here $\mu_0 = 0.20$ (another interpretation of the formulation in the problem is that $(1 + \mu_0)V_0 = 0.20$) and $V_0 = 100$. We get

$$\lambda_1 = 197.5 \text{ and } \lambda_2 = -21.375.$$

It follows that

$$\begin{aligned} w &= 395 \begin{bmatrix} 16.67 & -2.778 \\ -2.778 & 11.57 \end{bmatrix} \begin{bmatrix} 0.10 \\ 0.15 \end{bmatrix} \\ &\quad +(-42.75) \begin{bmatrix} 16.67 & -2.778 \\ -2.778 & 11.57 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -100 \\ 200 \end{bmatrix} \end{aligned}$$

**Remark.** There is a quicker way to arrive at the solution here. There are two constraints, and two free variables. The feasible set only consists of one point, namely the portfolio determined by

$$\begin{aligned} w^T \mathbf{1} &= V_0 \\ w^T \mu &= \mu_0 V_0 \end{aligned}$$

and the solution to this system of equations is

$$w = \begin{bmatrix} -100 \\ 200. \end{bmatrix}$$

(d) In this case the problem is

$$\left[\begin{array}{ll} \min & \frac{1}{2} w^T \Sigma w \\ \text{s.t.} & w_0 r_0 + w^T \mu = \mu_0 V_0 \\ & w_0 + w^T \mathbf{1} = V_0, \end{array}\right.$$

and the Lagrangian is

$$L = \frac{1}{2} w^T \Sigma w + \lambda_1 (\mu_0 V_0 - w_0 r_0 - w^T \mu) + \lambda_2 (V_0 - w_0 - w^T \mathbf{1}).$$

The first order conditions are

$$\begin{aligned} \nabla L &= \Sigma w - \lambda_1 \mu - \lambda_2 \mathbf{1} = 0 \\ \frac{\partial L}{\partial w_0} &= -\lambda_1 r_0 - \lambda_2 = 0 \qquad (\star) \\ \frac{\partial L}{\partial \lambda_1} &= \mu_0 V_0 - w_0 r_0 - w^T \mu = 0 \\ \frac{\partial L}{\partial \lambda_2} &= V_0 - w_0 - w^T \mathbf{1} = 0 \qquad (\star\star) \end{aligned}$$

We get

$$w = \lambda_1 \Sigma^{-1} \mu + \lambda_2 \Sigma^{-1} \mathbf{1}.$$

Using $(\star)$ we can write

$$w = \lambda_1 \Sigma^{-1} (\mu - r_0 \mathbf{1}).$$

Inserting this in $(\star\star)$ and $(\star\star\star)$ yields

$$\begin{bmatrix} 1 & \mathbf{1}^T \Sigma^{-1} (\mu - r_0 \mathbf{1}) \\ r_0 & \mu^T \Sigma^{-1} (\mu - r_0 \mathbf{1}) \end{bmatrix} \begin{bmatrix} w_0 \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} V_0 \\ \mu_0 V_0 \end{bmatrix}.$$

Using the given numerical values we get

$$w_0 = -82.1 \text{ and } \lambda_1 = 115.7,$$

and with this $\lambda_1$

$$w = \begin{bmatrix} 64.3 \\ 117.9 \end{bmatrix}.$$

**Problem 5**

(a) For a random variable $X$ and $\alpha \in (0, 1)$ we have

$$\mathrm{ES}_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha \mathrm{VaR}_u(X) du.$$

(b) $\mathrm{VaR}_\alpha(X)$ is defined by

$$\mathrm{VaR}_\alpha(X) = \min\{m \,|\, P(mR_0 + X < 0) \le \alpha\}.$$

Here $R_0 = 1$ and $X$ has a continuous distribution, so

$$P(mR_0 + X < 0) = P(X < -m) = P(X \le -m) = F_X(-m).$$

If $x < 0$, then

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt = \int_{-\infty}^{x} \frac{1}{2a} e^{t/a} dt = \frac{1}{2a} \left[ae^{t/a}\right]_{-\infty}^{x} = \frac{1}{2} e^{x/a},$$

and if $x \ge 0$

$$F_X(x) = \int_{-\infty}^{0} f_X(t)dt + \int_{0}^{x} f_X(t)dt = \frac{1}{2} + \frac{1}{2a} \left[-ae^{-t/a}\right]_{0}^{x} = 1 - \frac{1}{2} e^{-x/a}.$$

$F_X$ is a strictly increasing function, and thus for every $\alpha \in (0,1)$ $\mathrm{VaR}_\alpha(X)$ satisfies

$$\alpha = F_X(-\mathrm{VaR}_\alpha).$$

To calculate $\mathrm{ES}_{0.05}$ we need the Value-at-risk for $\alpha \in (0, 0.05]$, and this means that we need the distribution function for $x < 0$. It follows that

$$\alpha = \frac{1}{2} e^{-\mathrm{VaR}_\alpha(x)/a}$$

for any $\alpha \in (0, 1/2]$, and from this that

$$\mathrm{VaR}_\alpha(X) = -a \ln(2\alpha).$$

We get

$$\begin{aligned}
\mathrm{ES}_{0.05}(X) &= \frac{1}{0.05} \int_0^{0.05} \mathrm{VaR}_u du \\
&= 20 \int_0^{0.05} \big( -a \ln(2u) \big) du \\
&= -20a \int_0^{0.05} (\ln 2 + \ln u) du \\
&= -a \ln 2 - 20a \int_0^{0.05} \ln u \, du \\
&= \left\{ \int_0^x \ln t \, dt = x \ln x - x \text{ for } x > 0 \right\} \\
&= -a \ln 2 - 20a(0.05 \ln 0.05 - 0.05) \\
&= -a \ln 2 - a \log 0.05 + a \\
&= 3.303a.
\end{aligned}$$

Note that

$$\mathrm{ES}_{0.05}(X) = \mathrm{VaR}_{0.05}(X) + a.$$

(c) When $R_0 = 1$, the discounted loss $L = -X$ and we have

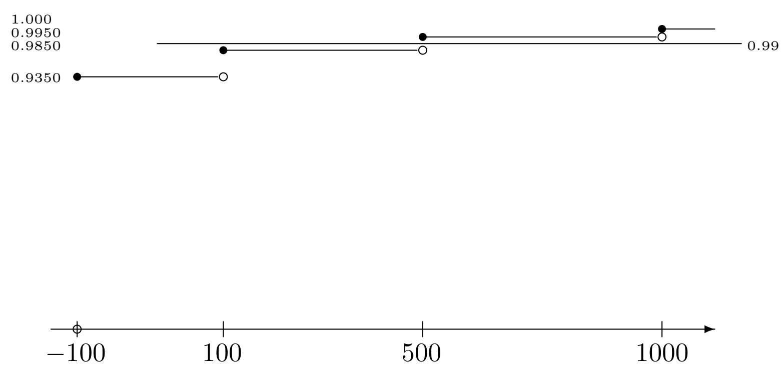
$$\mathrm{VaR}_{0.01}(X) = F_L^{-1}(1 - 0.01) = F_L^{-1}(0.99).$$

The loss has distribution function

$$L = \begin{cases} -100 & \text{with probability} \quad 0.9350 \\ 100 & \text{with probability} \quad 0.05 \\ 500 & \text{with probabiity} \quad 0.01 \\ 1\,000 & \text{with probability} \quad 0.005 \end{cases}$$

and inverse

$$F_L^{-1}(p) = \begin{cases} -100 & \text{if} \quad p \leq 0.9350 \\ 100 & \text{if} \quad 0.9350 < p \leq 0.9850 \\ 500 & \text{if} \quad 0.9859 < p \leq 0.9950 \\ 1000 & \text{if} \quad p > 0.9950 \end{cases}$$




We see that

$$\mathrm{VaR}_{0.01}(X) = F_L^{-1}(0.99) = 500.$$