SF2942 - PORTFOLIO THEORY AND RISK MANAGEMENT
SOLUTIONS TO FINAL EXAM, THURSDAY OCT 27 2016

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Allowed technical aids: calculator.

All answers must be carefully motivated. Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow. A correct answer with no or insufficient explanation will not receive full credit.

Every problem counts for a total of 10 points.

Good luck!

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Problem 1

Principal Component Analysis can be a useful tool in the context of immunization of cash flows. Suppose you have m hedging instruments to be used for the liability L and that the current zero-rate curve is described by $\mathbf{r}$. Furthermore, suppose that you use historical data to estimate the mean vector μ and covariance matrix Σ for changes in the zero-rate curve. Answer the following questions regarding PCA and immunization:

- a) What are the "principal components" and how do you determine them?
- b) How can you decide which and how many of the principal components to use?
- c) Can you give an example of when a PCA would *not* be particularly useful?
- d) Suppose that you have obtained proposed positions in the relevant hedging instruments, from which you can construct an immunization portfolio. Describe a way to try to evaluate the performance of the proposed portfolio using the estimates $\tilde{\mu}$ and $\tilde{\Sigma}$ of the mean vector and covariance matrix.

Motivate your answers properly.

Solution.

a) - b) See Section 3.6.1.

c) One such example is when there is no linear dependence between the components of $\Delta\mathbf{r}$. See e.g. the R markdown document from September 20.

d) See Section 3.6.1.

Problem 2

A company will pay out a a bonus to some of its employees, depending on whether they meet some individual performance goals or not, at the end of the year (time T). Each bonus is of size $c(S_T/S_0)$, $c > 0$, where S_t, is the spot price at time t of a share of the parent company's stock. Assume that the spot price follows Black's model: Current spot price is S_0 and we assume that S_T/S_0 has a lognormal distribution with parameters μ, σ.

The (known) number of employees who are eligible for a bonus is N and the probability that an employee meets his or her individual performance goals is estimated to be p; employees meet their goals independent of each other and of the parent company's share price.

The company wants to construct a hedge against the liability caused by the bonus system. The available instruments are:

- A zero-coupon bond with face value 1, current price B_0 and maturity at T.
- Shares in the stock of the parent company.

Find the optimal quadratic hedge of the liability at time T and compute the hedging error.

Solution. Let n be the random number of employees that receive a bonus; n has a binomial distribution with parameters N and p. The company's liability is then given by

$$L = nc\frac{S_T}{S_0}.$$

Let $Z = S_T$ be the future spot price of a share in the stock of the parent company. Our hedge is characterized by a pair (h_0, h), where h_0 is the number of positions we take in the zero-coupon bond and h the number of shares in the parent company stock; the time T value of the portfolio is

$$h_0 + hS_T.$$

The optimal quadratic hedge is given by

$$h = \frac{\mathrm{Cov}(L, S_T)}{\mathrm{Var}(S_T)}.$$

With the specific form of L we have

$$\begin{aligned}\mathrm{Cov}(L, S_T) &= \mathrm{Cov}\left(nc\frac{S_T}{S_0}, S_T\right)\\ &= \frac{c}{S_0}\mathrm{Cov}(nS_T, S_T).\end{aligned}$$

Using the assumed independence between n and S_T,

$$\mathrm{Cov}(nS_T, S_T) = E[n]\mathrm{Var}(S_T),$$

and it follows that the optimal hedge is given by, since $E[n] = Np$,

$$h = \frac{c}{S_0}Np,$$

with the corresponding position in the zero-coupon bond being

$$\begin{aligned} h_0 &= E\left[\frac{c}{S_0} n S_T\right] - hE\left[S_T\right] \\ &= \frac{c}{S_0} E[n]E[S_T] - \frac{c}{S_0} E[n]E[S_T] \\ &= 0. \end{aligned}$$

The hedging error for the optimal quadratic hedge is

$$L - hS_T = c\frac{S_T}{S_0}\left(n - Np\right).$$

That is, the risk is only due to the uncertainty in the number of bonuses to be paid out (n vs. the expected number Np).

Problem 3

Consider a time period from 0 to $T > 0$ and suppose there is a risk-free asset with return R_0 over that period. Suppose that you are investing the known initial capital V_0 at time 0 in a portfolio P that has random future value V at time T.

a) Define the risk measures *Value-at-Risk* and *Expected shortfall* in terms of V_0, V and R_0.

b) Suppose that you are comparing the portfolio P to an alternative portfolio $\tilde{P}$, with future value $\tilde{V}$ (same initial capital used). You know that the Value-at-Risk associated with the portfolio P and the Expected Shortfall associated with $\tilde{P}$, both at level $p = 0.01$, equal some $C > 0$. If you want to use Expected Shortfall to choose a portfolio - less risk is to be preferred - which of the two portfolios would you prefer? Does this answer depend on the (assumed) properties of the distributions of V and $\tilde{V}$? Be as precise as you can.

Solution.

a) Value-at-Risk at level $p \in (0,1)$, $\mathrm{VaR}_p(V - V_0R_0)$, is defined as

$$\mathrm{VaR}_p(V - V_0R_0) = \min\left\{m : P(mR_0 + V - V_0R_0 < 0) \leq p\right\} = \min\left\{m : P(R_0(m - V_0) + V < 0) \leq p\right\}$$

Alternatively, we can define $L = -(V - V_0R_0)/R_0 = V_0 - V/R_0$ and express Value-at-Risk as

$$\begin{aligned} \mathrm{VaR}_p(V - V_0R_0) &= \min\left\{F_L(m) \geq 1 - p\right\} \\ &= F_L^{-1}(1-p). \end{aligned}$$

where F_L is the distribution function of L.

Having defined $\mathrm{VaR}_p(V - V_0R_0)$, Expected Shortfall (at level p) is defined as

$$\begin{aligned} \mathrm{ES}_p(V - V_0R_0) &= \frac{1}{p}\int_0^p \mathrm{VaR}_u(V - V_0R_0)du \\ &= \frac{1}{p}\int_{1-p}^1 F_L^{-1}(u)du. \end{aligned}$$

b) Let $X = V - V_0 R_0$, $L = -X/R_0$ and define $\tilde{X}$, $\tilde{L}$ analogously for the portfolio $\tilde{P}$. We are given the information that

$$\mathrm{VaR}_p(X) = \mathrm{ES}_p(\tilde{X}) = c.$$

Ideally, to compare the two portfolios we would compute $\mathrm{ES}_p(X)$. However, without knowing more about the distribution of V we cannot compute $\mathrm{VaR}_u(X)$ for $u \in (0,p)$, and thus not $\mathrm{ES}_p(X)$. However, we can use the given information and the fact that $\mathrm{ES}_p(X)$ is defined as

$$\mathrm{ES}_p(X) = \frac{1}{p}\int_0^p \mathrm{VaR}_u(X)du.$$

Regardless of the distribution of X, $\mathrm{VaR}_p(X)$ is non-increasing in p. Thus, we have that

$$\mathrm{VaR}_u(X) \geq \mathrm{VaR}_p(X) = C \quad \text{for } u \in (0,p).$$

It follows that

$$\mathrm{ES}_p(X) \geq \frac{1}{p}\int_0^p C du = C.$$

If the distribution function of X is strictly increasing on at least some part of the interval $(\mathrm{VaR}_p(X), \infty)$, then the inequality in the last display is strict. We conclude that regardless of the distribution of V,

$$\mathrm{ES}_p(X) \geq \mathrm{ES}_p(\tilde{X}),$$

and from a risk perspective, based on expected shortfall, the portfolio $\tilde{P}$ is to be preferred.

Problem 4

Consider a time period of length $T > 0$ and suppose that there is a risk-free asset with return R_0 and n risky assets with random returns $R_1, \ldots, R_n$ to invest in. A portfolio can be described by the corresponding monetary portfolio weights w_0 and $\mathbf{w} = (w_1, \ldots w_n)^{\mathsf{T}}$.

An investor with initial capital V_0 has previously chosen her portfolio according to the investment problem

$$\begin{aligned} &\text{maximize} \quad E[V] - \frac{c}{2V_0}\mathrm{Var}(V), \\ &\text{subject to} \quad w_0 + \mathbf{w}^{\mathsf{T}}\mathbf{1} \leq V_0, \end{aligned}$$

where $V = w_0 R_0 + \mathbf{w}^{\mathsf{T}}\mathbf{R}$ and $c > 0$ is a trade-off parameter reflecting the investor's attitude towards risk.

a) Suppose $R_0 = 1.03$ and the risky assets the investor is thinking of investing in are two defaultable bonds. The face value of both bonds are \$1000000 and this is paid to the holder if the issuer of the bond does not default before time T. The current prices are given by

$$P_k = \frac{1000000}{R_0}(1 - p_k), \quad k = 1, 2,$$

where $p_1 = 1/6$ and $p_2 = 1/5$ can be thought of as the market implied probabilities of default. The issuers of the bonds belong to the same sector and can not be considered to default independently of each other. The investor believes that the default probabilities are overestimated and her subjective probability of default is instead $q = 1/10$ for both bonds. Moreover, her subjective view is that the conditional probability that one bond defaults, given that the other has already done so, is $(1+q)/2$. The investor is only interested in taking long positions in the two bonds. Find the optimal portfolio under these conditions.

b) Consider again the general setting with n risky assets. As a measure of the performance of *any* portfolio $\mathbf{w}$ we can define the *information ratio* $IR(w_0, \mathbf{w})$ by

$$IR(w_0, \mathbf{w}) = \frac{E\left[w_0 R_0 + \mathbf{w}^\intercal \mathbf{R}\right]}{\sqrt{\mathrm{Var}(w_0 R_0 + \mathbf{w}^\intercal \mathbf{R})}}.$$

Suppose the investor is considering choosing her portfolio according to maximizing $IR(w_0, \mathbf{w})$, amongst affordable portfolios. Find the portfolio she would choose if the decision is based on the information ratio rather than the investment problem.
Addition: Describe the portfolio choice for a given (arbitrary) level of risk.

Solution.
a) Let I_1 and I_2 be default indicators for the two bond issuers. I_1 and I_2 both take the value 1 with (subjective) probability $q = 1/10$. Let R_1 and R_2 denote the returns of the two bonds:

$$R_k = \frac{10^6}{P_k}(1 - I_k) = \frac{R_0}{1 - p_k}(1 - I_k), \quad k = 1, 2.$$

In order to solve the investment problem of interest we need the mean vector and covariance matrix associated with the return vector $\mathbf{R}^\intercal = (R_1, R_2)$.

The expected value of the returns are

$$\mu_k = E[R_k] = \frac{R_0}{1 - p_k}(1 - q), \quad k = 1, 2,$$

which gives the mean vector. Similarly, the variances are

$$\mathrm{Var}(R_k) = \frac{R_0^2}{(1 - p_k)^2}\mathrm{Var}(1 - I_k) = \frac{R_0^2}{(1 - p_k)^2} q(1 - q), \quad k = 1, 2.$$

Remains to determine the covariance:

$$\begin{aligned}
\mathrm{Cov}(R_1, R_2) &= \mathrm{Cov}\left(\frac{R_0}{1 - p_1}(1 - I_1), \frac{R_0}{1 - p_2}(1 - I_2)\right) \\
&= \frac{R_0^2}{(1 - p_1)(1 - p_2)}\mathrm{Cov}(1 - I_1, 1 - I_2) \\
&= \frac{R_0^2}{(1 - p_1)(1 - p_2)}\mathrm{Cov}(I_1, I_2).
\end{aligned}$$

The covariance term in the last equation can be expressed as

$$\mathrm{Cov}(I_1, I_2) = E[I_1 I_2] - q^2 = P(I_1 = 1, I_2 = 1) - q^2.$$

The investor's subjective view is that

$$P(I_1 = 1 | I_2 = 1) = P(I_2 = 1 | I_1 = 1) = \frac{1+q}{2}.$$

It follows that

$$\begin{aligned} P(I_1 = 1, I_2 = 1) &= P(I_1 = 1 | I_2 = 1) P(I_2 = 1) \\ &= \frac{q(1+q)}{2}. \end{aligned}$$

It follows that the covariance between the returns is

$$\begin{aligned} \mathrm{Cov}(R_1, R_2) &= \frac{R_0^2}{(1-p_1)(1-p_2)} \mathrm{Cov}(I_1, I_2) \\ &= \frac{R_0^2}{(1-p_1)(1-p_2)} \left(\frac{q+q^2}{2} - q^2 \right) \\ &= \frac{R_0^2}{(1-p_1)(1-p_2)} \left(\frac{q-q^2}{2} \right) \end{aligned}$$

In particular, this implies that the correlation between the returns is

$$\begin{aligned} \mathrm{Cor}(R_1, R_2) &= \frac{\mathrm{Cov}(R_1, R_2)}{\sqrt{\mathrm{Var}(R_1)\mathrm{Var}(R_2)}} \\ &= \frac{1}{2}. \end{aligned}$$

If we use ρ to denote the correlation, this gives the covariance matrix Σ

$$\Sigma = \frac{R_0^2 q(1-q)}{(1-p_1)(1-p_2)} \begin{pmatrix} \frac{1-p_2}{1-p_1} & \rho \\ \rho & \frac{1-p_1}{1-p_2} \end{pmatrix},$$

which has inverse

$$\Sigma^{-1} = \frac{(1-p_1)(1-p_2)}{R_0^2 q(1-q)(1-\rho^2)} \begin{pmatrix} \frac{1-p_1}{1-p_2} & -\rho \\ -\rho & \frac{1-p_2}{1-p_1} \end{pmatrix}.$$

With the mean vector $\boldsymbol{\mu}$ and Σ available we can write down the solution $\mathbf{w}$ to the trade-off problem for the given c. Ignoring any restrictions regarding short positions, the optimal weights are

$$\begin{aligned} \mathbf{w} &= \frac{V_0}{c} \Sigma^{-1} (\boldsymbol{\mu} - \mathbf{1} R_0) \\ &= \frac{V_0}{c} \frac{(1-p_1)(1-p_2)}{R_0^2 q(1-q)(1-\rho^2)} \begin{pmatrix} \frac{1-p_1}{1-p_2}\mu_1 - \rho\mu_2 + R_0 \left(\rho - \frac{1-p_1}{1-p_2} \right) \\ -\rho\mu_1 + \frac{1-p_2}{1-p_1}\mu_2 + R_0 \left(\rho - \frac{1-p_2}{1-p_1} \right) \end{pmatrix}. \end{aligned}$$

If the components are both non-negative this solution satisfies the condition of only long positions and we have found the sought-after portfolio. Inserting numerical values,

$$\boldsymbol{\mu}^{\mathsf{T}} = (1.1124, 1.1588),$$

and

$$\frac{1-p_1}{1-p_2}\mu_1 - \rho\mu_2 + R_0\left(\rho - \frac{1-p_1}{1-p_2}\right) = 0.02146,$$
$$-\rho\mu_1 + \frac{1-p_2}{1-p_1}\mu_2 + R_0\left(\rho - \frac{1-p_2}{1-p_1}\right) = 0.08240.$$

We conclude that the solution to the trade-off problem satisfies the "long positions only"-condition. Inserting the remaining numerical values we can express the optimal weights as

$$\mathbf{w} = \frac{V_0}{c}\begin{pmatrix}0.1998\\0.7671\end{pmatrix}.$$

b) In the mean-variance setting we quantify risk by the variance of the portfolio value. A given level of risk thus corresponds to fixing the variance σ^2 of the portfolio value. To maximize $IR(w_0, \mathbf{w})$ for a given level of risk we need to solve

$$\begin{aligned} \text{maximize} \quad & E[V], \\ \text{subject to} \quad & w_0 + \mathbf{w}^\intercal \mathbf{1} \le V_0, \\ & \mathbf{w}^\intercal \Sigma \mathbf{w} = \sigma^2 \end{aligned}$$

This is a version of the maximization-of-expectation problem with equality constraint for the variance. It is not difficult to verify that it has as optimal solution

$$\mathbf{w} = \sigma \frac{\Sigma^{-1}(\boldsymbol{\mu} - R_0\mathbf{1})}{\sqrt{(\boldsymbol{\mu} - R_0\mathbf{1})^\intercal \Sigma^{-1}(\boldsymbol{\mu} - R_0\mathbf{1})}},$$

where $\boldsymbol{\mu}$ and Σ is the mean vector and covariance matrix, respectively, of the underlying return vector. This is the portfolio the investor would choose for a given level of risk (reflected in σ). The corresponding expected return is

$$w_0 R_0 + \mathbf{w}^\intercal \boldsymbol{\mu} = V_0 R_0 + \sigma\sqrt{(\boldsymbol{\mu} - R_0\mathbf{1})^\intercal \Sigma^{-1}(\boldsymbol{\mu} - R_0\mathbf{1})},$$

and the information ratio is

$$\frac{V_0 R_0}{\sigma} + \sqrt{(\boldsymbol{\mu} - R_0\mathbf{1})^\intercal \Sigma^{-1}(\boldsymbol{\mu} - R_0\mathbf{1})}$$

Note that the optimal portfolio is the same as that which solves the trade-off problem if we set

$$\sigma = \frac{V_0}{c}\sqrt{(\boldsymbol{\mu} - R_0\mathbf{1})^\intercal \Sigma^{-1}(\boldsymbol{\mu} - R_0\mathbf{1})}.$$

An alternative answer goes as follows:
Recall the notion of the efficient frontier: For a given c, we can compute the optimal portfolio weights $(w_0^*(c), \mathbf{w}^*(c))$ in the trade-off problem and corresponding expected return and variance,

$$\mu^*(c) = (\mathbf{w}^*(c))^\intercal \boldsymbol{\mu}, \quad \sigma^*(c) = \sqrt{(\mathbf{w}^*(c))^\intercal \Sigma \mathbf{w}^*(c)}.$$

Because $\mathbf{w}^*(c)$ is the optimal vector of weights for a given c, for any suboptimal solution $\mathbf{w}$ to the trade-off problem, the corresponding mean and standard deviation will be a point in the $\mu - \sigma$-plane that lies below the efficient frontier. That is, for a given value of c, no suboptimal solution to the trade-off problem will lie above the efficient frontier in the $\sigma - \mu$-plane. This implies that the ratio between expected return and standard deviation for any suboptimal portfolio will be less than the same ratio for $\mathbf{w}^*(c)$, for some c. Thus, for any c, for any other portfolio with standard deviation $\sigma^*(c)$ and expected return μ we have

$$\frac{\mu^*(c)}{\sigma^*(c)} > \frac{\mu}{\sigma^*(c)}.$$

That is the weights $(w_0, \mathbf{w}^*(c))$ maximizes the information ratio given that the risk is $\sigma^*(c)$.

Lastly, the following answer will also yield some partial credit:
With no restriction on the expected return of the portfolio or the variance, the optimal choice is to take the position $w_0 = V_0$, and $\mathbf{w}$ the zero vector in $\mathbb{R}^n$. Indeed, this portfolio has (expected) return $V_0 R_0$ and zero variance, hence the information ratio is infinite. This can also be deduced from solving the trade-off problem and computing the corresponding information ratio:

$$\frac{cR_0}{\sqrt{(\boldsymbol{\mu} - R_0\mathbf{1})^\intercal \Sigma^{-1} (\boldsymbol{\mu} - R_0\mathbf{1})}} + \sqrt{(\boldsymbol{\mu} - R_0\mathbf{1})^\intercal \Sigma^{-1} (\boldsymbol{\mu} - R_0\mathbf{1})},$$

which can be made arbitrarily large by sending c to infinity. This amounts to an investor that is becoming more and more risk averse, eventually placing no value in the possible reward from investing in the risky assets.

Problem 5

A company is considering buying a full coverage insurance, for the duration of one year, for some of its property. The company is worried about the loss in value of its property due to accident - for example due to extreme weather - and with insurance the value is fully restored. You can assume that there is no other causes for the value to be reduced during the year. The company can be considered a utility-maximizer with utility function $u(x) = x^\gamma$, $x > 0$.

a) What are possible values for γ if the company is to be considered risk-averse?

b) What is the coefficient of absolute risk aversion for the company?

c) In the case of an accident the value of the company's property is estimated to be reduced by a factor 1/2 and the probability of an accident is estimated to be some $p \in (0, 1)$. Express in terms of γ and p what the company is willing to pay for a full coverage insurance under these assumptions.

d) Suppose now that the type of accidents causing damage to the company's property are categorized as two types. Given that there is an accident, the first type has probability q_1 and causes an estimated loss of 15% in property value, whereas the second type has probability q_2, with $q_1 + q_2 = 1$, and causes a loss of $100U$ % of the property value, where U has a uniform distribution on $[\frac{1}{2}, 1]$. Under these assumptions, what is the company willing to pay for the insurance?

Solution.
a) Risk-averse behavior corresponds to concave utility functions. Thus, possible values for γ are $\gamma \in (0, 1]$.

b) The coefficient of absolute risk aversion is defined as

$$A(x) = -\frac{u''(x)}{u'(x)}.$$

For the given u this amounts to

$$\begin{aligned} A(x) &= -\frac{\gamma(\gamma-1)x^{\gamma-2}}{\gamma x^{\gamma-1}} \\ &= \frac{1-\gamma}{x}. \end{aligned}$$

c) Let V denote the initial value of the company's property and V_1 the value in one year. If I is the indicator of whether or not an accident has occurred during the year - $I = 1$ means there has been an accident - then the value in one year without insurance is

$$V_1 = V(1 - \frac{1}{2}I).$$

If the company buys insurance for the price cV, some c, the value in one year is instead the deterministic amount

$$V_1 = V(1-c).$$

The expected utility in the two cases are

$$\begin{aligned} E\left[u(V_1)\right] &= V^\gamma(1-p) + V^\gamma 2^{-\gamma} p \\ &= V^\gamma(1 - p + 2^{-\gamma}p), \end{aligned}$$

with insurance and

$$V^\gamma(1-c)^\gamma.$$

without insurance. The company would thus buy the insurance if

$$V^\gamma(1 - p + 2^{-\gamma}p) \leq V^\gamma(1-c)^\gamma,$$

which is equivalent to

$$c \leq 1 - \left(1 - p + 2^{-\gamma}p\right)^{1/\gamma}.$$

That is the absolute premium (i.e., not expressed as a fraction of V) can be at most

$$V\left(1 - \left(1 - p + 2^{-\gamma}p\right)^{1/\gamma}\right).$$

d) Using the same notation as in (c), the value in one year without insurance is now

$$V_1 = V(1 - I_1 - I_2) + V(1-\kappa)I_1 + V(1-U)I_2,$$

where $\kappa = 0.15$, U is a uniform random variable on $[\frac{1}{2}, 1]$ and I_1 and I_2 are random variables indicating whether an accident is of the first or second type (both are 0 if no accident has occurred). Since an accident can only be of one type, at most one of I_1 and I_2 can take the value 1; we have

$$P(I_1 = 1 | I_1 + I_2 = 1) = q_1, \quad P(I_2 = 1 | I_1 + I_2 = 1) = q_2.$$

Note that the conditioning is needed because we only know the conditional probability of either type of accident given that an accident has occurred. Using the law of total probability we can compute the unconditional probability of $\{I_1 = 1\}$ as

$$\begin{aligned} P(I_1 = 1) &= P(I_1 = 1 | I_1 + I_2 = 1)P(I_1 + I_2 = 1) + P(I_1 = 1 | I_1 + I_2 = 0)P(I_1 + I_2 = 0) \\ &= q_1 p. \end{aligned}$$

Similarly,

$$P(I_2 = 1) = q_2 p.$$

The expected utility without insurance becomes

$$E[u(V_1)] = E[V^\gamma I\{I_1 = I_2 = 0\}] + E[(V(1-\kappa))^\gamma I\{I_1 = 1\}] + E[(V(1-U))^\gamma I\{I_2 = 1\}].$$

The first two are simple in the sense that it is only the indicator function in each of them that is a random variable. We have that

$$E[V^\gamma I\{I_1 = I_2 = 0\}] = V^\gamma P(I_1 = I_2 = 0) = V^\gamma (1-p),$$

and

$$E[(V(1-\kappa))^\gamma I\{I_1 = 1\}] = (V(1-\kappa))^\gamma P(I_1 = 1) = V^\gamma (1-\kappa)^\gamma p q_1.$$

For the third term we use that the random variable U is independent of I_2:

$$E[(V(1-U))^\gamma I\{I_2 = 1\}] = V^\gamma E[(1-U)^\gamma] P(I_2 = 1) = V^\gamma E[(1-U)^\gamma] p q_2.$$

The remaining expected value is

$$\begin{aligned} E[(1-U)^\gamma] &= \int_{1/2}^{1} (1-u)^\gamma 2 du \\ &= \left[-\frac{2}{\gamma+1}(1-u)^{\gamma+1} \right]_{1/2}^{1} \\ &= \frac{2^{-\gamma}}{\gamma+1} \end{aligned}$$

Combining the three terms, the expected utility without insurance is

$$E[u(V_1)] = V^\gamma \left(1 - p + (1-\kappa)^\gamma p q_1 + \frac{2^{-\gamma}}{\gamma+1} p q_2 \right)$$

The expected utility with insurance is the same as in (c) and the relative premium c should thus be such that

$$V^\gamma\left(1-p+(1-\kappa)^\gamma pq_1+\frac{2^{-\gamma}}{\gamma+1}pq_2\right)\le V^\gamma(1-c)^\gamma.$$

This is equivalent to

$$c\le 1-\left(1-p+(1-\kappa)^\gamma pq_1+\frac{2^{-\gamma}}{\gamma+1}pq_2\right)^{1/\gamma},$$

and the absolute premium can be at most

$$V\left(1-\left(1-p+(1-\kappa)^\gamma pq_1+\frac{2^{-\gamma}}{\gamma+1}pq_2\right)^{1/\gamma}\right).$$

This is the highest (absolute) price the company would be willing to pay for the insurance.