RISK AND PORTFOLIO ANALYSIS: SOLUTIONS TO EXERCISES IN CHAPTERS 1-6

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ABSTRACT. The current collection of solutions to the exercises in the first part of the book *Risk and Portfolio Analysis: principles and methods* is not yet fully complete. Please inform us if you spot any errors.

CHAPTER 1

Exercise 1.1. *(Arbitrage in bond prices)*

(a) The cash flow of Bond D can be generated by the portfolio consisting of $106/200$ units of Bond C, $6/102$ units of Bond B and $(6-12/102)/100$ units of Bond A. The price of the portfolio is

$$0.53186.2 \cdot 98.51 + \frac{6}{102} \cdot 100.71 + 0.53 \cdot 188.03 = 111.3747,$$

whereas the price of Bond D is 111.55. Thus, short selling Bond D and buying the above bond portfolio create a profit of \$0.1753

(b) The cash flow of Bond A implies $d_1 = 0.9851$. The cash flow of bond D and linear interpolation of discount factors provide the equations

$$111.55 = 6d_1 + 6d_2 + 106d_3,$$
$$d_2 = d_1 + \frac{d_3 - d_1}{t_3 - t_1}(t_2 - t_1)$$

that can be solved for d_2 and d_3. The result is $d_2 = 0.9636$ and $d_3 = 0.9421$. Finally, the cash flow of Bond E provides the equation

$$198.96 = 4d_1 + 4d_2 + 4d_3 + 204d_4$$

which implies $d_4 = 0.9186$. By Theorem 1.1 (ii) we have proved the absence of arbitrage.

Exercise 1.2. *(Put-call parity)*

(a) The payoff function f can be written as $f(x) = x + (K_1 - x)_+ - (x - K_2)_+$. Replacing x by S_T and takeing expectation with respect to the forward probability Q yields the collar forward price

$$\begin{aligned} \mathrm{E}_Q[f(S_T)] &= \mathrm{E}_Q[S_T + (K_1 - S_T)_+ - (S_T - K_2)_+] \\ &= \mathrm{E}_Q[S_T] + \mathrm{E}_Q[(K_1 - S_T)_+] - \mathrm{E}_Q[(S_T - K_2)_+], \end{aligned}$$

where the three expectations above, in the order they appear, correspond to the forward price of the underlying asset, of the put option payoff and on the call option payoff.

(b) The payoff function of the risk reversal is $g(x) = (x - A)_+ - (B - x)_+$. If S_0 denotes the current spot price, then the statement that both options are out of the money means that

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$S_0 < A$ and $S_0 > B$, i.e. $B < A$. Set $K_1 = B$ and $K_2 = A$ and notice that (draw the figure!) $f(x) = x - g(x)$. In particular, the forward prices are related as

$$\mathrm{E}_Q[g(S_T)] = \mathrm{E}_Q[S_T] - \mathrm{E}_Q[f(S_T)].$$

Exercise 1.3. *(Sports betting)*

The best available odds are 4.70 on 'Everton', 3.70 on 'draw', and 1.95 on 'Manchester City'. Using these odds and betting 213 on 'Everton', 271 on 'draw' and 513 on 'Manchester City' is an arbitrage opportunity. Indeed, placing these bets costs $213 + 271 + 513 = 997$ and pays

$$\begin{aligned}
&213 \cdot 4.70 = 1001.1 \text{ if the outcome is 'Everton'}\\
&271 \cdot 3.70 = 1002.7 \text{ if the outcome is 'draw', and}\\
&513 \cdot 1.95 = 1000.35 \text{ if the outcome is 'Manchester City'.}
\end{aligned}$$

Exercise 1.4. *(Lognormal model)*

(a) Straightforward computations give

$$\begin{aligned}
\mathrm{E}[e^{aZ} I\{Z > b\}] &= \int_b^\infty e^{az} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz = e^{a^2/2} \int_b^\infty \frac{e^{-(z-a)^2/2}}{\sqrt{2\pi}} dz = e^{a^2/2} \int_{b-a}^\infty \frac{e^{-w^2/2}}{\sqrt{2\pi}} dw\\
&= e^{a^2/2} \Phi(a - b),
\end{aligned}$$

where in the last step we used the relation $\Phi(x) = 1 - \Phi(-x)$.

(b) From the result in (a) we obtain

$$\begin{aligned}
\mathrm{E}[(R-c)_+] &= \mathrm{E}\Big[(e^{\mu+\sigma Z} - c) I\Big\{Z > \frac{\log c - \mu}{\sigma}\Big\}\Big]\\
&= e^\mu \,\mathrm{E}\Big[e^{\sigma Z} I\Big\{Z > \frac{\log c - \mu}{\sigma}\Big\}\Big] - c\Phi\Big(\frac{\mu - \log c}{\sigma}\Big)\\
&= e^{\mu + \sigma^2/2} \Phi\Big(\sigma + \frac{\mu - \log c}{\sigma}\Big) - c\Phi\Big(\frac{\mu - \log c}{\sigma}\Big).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{E}[(R-c)_+^2] &= \mathrm{E}\Big[(e^{2\mu+2\sigma Z} - 2ce^{\mu+\sigma Z} + c^2) I\Big\{Z > \frac{\log c - \mu}{\sigma}\Big\}\Big]\\
&= e^{2\mu+2\sigma^2} \Phi\Big(2\sigma + \frac{\mu - \log c}{\sigma}\Big)\\
&\quad - 2ce^{\mu+\sigma^2/2} \Phi\Big(\sigma + \frac{\mu - \log c}{\sigma}\Big) + c^2 \Phi\Big(\frac{\mu - \log c}{\sigma}\Big).
\end{aligned}$$

Combining the expressions for $\mathrm{E}[(R-c)_+]$ and $\mathrm{E}[(R-c)_+^2]$ gives

$$\begin{aligned}
\mathrm{Var}((R-c)_+) &= \mathrm{E}[(R-c)_+^2] - (\mathrm{E}[(R-c)_+])^2\\
&= e^{2\mu+2\sigma^2} \Phi\Big(2\sigma + \frac{\mu - \log c}{\sigma}\Big)\\
&\quad - 2ce^{\mu+\sigma^2/2} \Phi\Big(\sigma + \frac{\mu - \log c}{\sigma}\Big) + c^2 \Phi\Big(\frac{\mu - \log c}{\sigma}\Big)\\
&\quad - \Big(e^{\mu+\sigma^2/2} \Phi\Big(\sigma + \frac{\mu - \log c}{\sigma}\Big) - c\Phi\Big(\frac{\mu - \log c}{\sigma}\Big)\Big)^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{E}[R(R-c)_+] &= \mathrm{E}\Big[(e^{2\mu+2\sigma Z} - ce^{\mu+\sigma Z})I\Big\{Z > \frac{\log c - \mu}{\sigma}\Big\}\Big] \\
&= e^{2\mu+2\sigma^2}\Phi\Big(2\sigma + \frac{\mu - \log c}{\sigma}\Big) - ce^{\mu+\sigma^2/2}\Phi\Big(\sigma + \frac{\mu - \log c}{\sigma}\Big)
\end{aligned}$$

which leads to

$$\begin{aligned}
\mathrm{Cov}(R,(R-c)_+) &= \mathrm{E}[R(R-c)_+] - \mathrm{E}[R]\,\mathrm{E}[(R-c)_+] \\
&= e^{2\mu+2\sigma^2}\Phi\Big(2\sigma + \frac{\mu - \log c}{\sigma}\Big) - ce^{\mu+\sigma^2/2}\Phi\Big(\sigma + \frac{\mu - \log c}{\sigma}\Big) \\
&\quad - e^{\mu+\sigma^2/2}\Big(e^{\mu+\sigma^2/2}\Phi(\sigma + \frac{\mu - \log c}{\sigma}) - c\Phi\Big(\frac{\mu - \log c}{\sigma}\Big)\Big).
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathrm{E}[(R-c)_+(R-d)_+] &= \mathrm{E}\Big[(R^2 - (c+d)R + cd)I\Big\{Z > \frac{\log d - \mu}{\sigma}\Big\}\Big] \\
&= e^{2\mu+2\sigma^2}\Phi\Big(2\sigma + \frac{\mu - \log d}{\sigma}\Big) - (c+d)e^{\mu+\sigma^2/2}\Phi\Big(\sigma + \frac{\mu - \log d}{\sigma}\Big) \\
&\quad + cd\Phi\Big(\frac{\mu - \log d}{\sigma}\Big).
\end{aligned}$$

This leads to

$$\begin{aligned}
\mathrm{Cov}((R-c)_+,(R-d)_+) &= \mathrm{E}[(R-c)_+(R-d)_+] - \mathrm{E}[(R-c)_+]\,\mathrm{E}[(R-d)_+] \\
&= e^{2\mu+2\sigma^2}\Phi\Big(2\sigma + \frac{\mu - \log d}{\sigma}\Big) \\
&\quad - (c+d)e^{\mu+\sigma^2/2}\Phi\Big(\sigma + \frac{\mu - \log d}{\sigma}\Big) \\
&\quad + cd\Phi\Big(\frac{\mu - \log d}{\sigma}\Big) \\
&\quad - \Big(e^{\mu+\sigma^2/2}\Phi\Big(\sigma + \frac{\mu - \log c}{\sigma}\Big) - c\Phi\Big(\frac{\mu - \log c}{\sigma}\Big)\Big) \\
&\quad\ \times \Big(e^{\mu+\sigma^2/2}\Phi\Big(\sigma + \frac{\mu - \log d}{\sigma}\Big) - d\Phi\Big(\frac{\mu - \log d}{\sigma}\Big)\Big)
\end{aligned}$$

Exercise 1.5. *(Risky bonds)*

(a) We obtain the risk-free rates from the cash flows of bonds A and B by solving

$$98 = 100e^{-r_1}, \quad 104 = 5e^{-r_1} + 105e^{-2r_2}.$$

The discount factors are $d_1 = e^{-r_1} = 0.98$ and $d_2 = e^{-2r_2} \approx 0.9438095$ and the zero rates are $r_1 \approx 0.02020271$ and $r_2 \approx 0.02891545$. The credit spreads are obtained by solving

$$93 = 100e^{-(r_1+s_1)}, \quad 98 = 10e^{-(r_1+s_1)} + 110e^{-2(r_2+s_2)}$$

for s_1 and s_2. We find that $s_1 \approx 0.05236799$ and $s_2 \approx 0.07869478$.

Let I_1 and I_2 denote the default indicators for the risky bank over a one- and two-year period, respectively. Denote the corresponding unknown default probabilities by q_1 and q_2 and solve

$$93 = 100e^{-r_1}(1-q_1), \quad 98 = 10e^{-r_1}(1-q_1) + 110e^{-2r_2}(1-q_2)$$

for q_1 and q_2. We find that $q_1 = 5/98 \approx 0.05102041$ and $q_2 \approx 0.1456288$.

(b) For the investor it only makes sense to invest in Bond C and Bond D (since the investor believes that they cannot default and, hence, are underpriced). Let $w \in [0, 10^4]$ be the amount invested in Bond C. Notice that at time 1 the investor invests any cash flow in Bond D which at that time is a 1-year bond with the random price $110e^{-(r+s)}$, where r and s are independent and normally distributed. Given the assumption of no defaults the cash flow at time 1 is

$$w\frac{100}{93} + (10^4 - w)\frac{10}{98}$$

and the random cash flow at time 2 is

$$w\frac{100}{93}e^{r+s} + (10^4 - w)\Big(\frac{10}{98}e^{r+s} + 110\Big).$$

Since $r+s$ is normally distributed with mean 0.16 and variance $0.01^2 + 0.03^2$ the expected value of e^{r+s} is $e^{0.1605}$. The expected cash flow at time 2 can be written $c_0 + c_1 w$ with $0 < c_1 \approx 0.02021585$. In particular, the expected cash flow is maximized by choosing $w = 10^4$: all money goes into Bond C.

(c) Here it is assumed that the investor has a correct view of how the market will price Bond D at time 1 but is wrong in assuming that the bonds are non-defaultable. The default probabilities provided by the market prices at time 0 are correct.

Investing everything in Bond C at time 0 and, at time 1, re-investing everything in Bond D gives that cash flow $10^4(1 - I_1)100/93$ at time 1 and $10^4(1 - I_2)e^{r+s}100/93$ at time 2, where r and s are independent and independent of the default indicators. Plot the corresponding distribution function and simulate from the distribution of (I_2, r, s) to produce a histogram illustration the distribution of the terminal value of the investors strategy under the above assumptions.

Chapter 3

Exercise 3.1. *(Annuity)*

(a) Let τ be the random year of death of the policy holder, where $\tau = 1$ means death of the policy holder within one year from today. The annuity contract gives the policy holder the cash flow $\{C_k\}_{k \geq 1}$, where

$$C_k = \begin{cases} 0 & k < y, \\ c & k \geq y, \tau > k, \\ 0 & k \geq y, \tau \leq k. \end{cases}$$

The value today V_0 of the annuity cash flow is

$$V_0 = \mathrm{E}\Big[\sum_{k=y}^{\infty} ce^{-kr_k} I\{\tau > k\}\Big] = c\sum_{k=y}^{\infty} e^{-kr_k}\,\mathrm{P}(\tau > k).$$

From the Gompertz-Makeham formula for the mortality rate, $\mu_0(x+u) = A + Re^{\alpha(x+u)}$, at age $x+u$ of an age-x policy holder we find that

$$\begin{aligned}\mathrm{P}(\tau > k) &= \exp\Big\{-\int_0^k \Big(A + Re^{\alpha(x+u)}du\Big)\Big\} \\ &= \exp\Big\{-Ak - \frac{Re^{\alpha x}}{\alpha}\big(e^{\alpha k} - 1\big)\Big\}.\end{aligned}$$

(b) Here, $c = 5000$, $y = 1$, $r_k = 0.04$ for all k, $x = 65$, $A = 0.002$, $R = e^{-12}$, and $\alpha = 0.12$. Inserting the numerical values into

$$V_0(n) = c\sum_{k=y}^{n} e^{-kr_k} \exp\Big\{-Ak - \frac{Re^{\alpha x}}{\alpha}\big(e^{\alpha k}-1\big)\Big\}.$$

gives $V_0(200) \approx \$51,067$ and $V_0(30) \approx \$51,051$. The corresponding values of the truncated, at $n = 200$ and $n = 30$, annuity payments to a hypothetical immortal policy holder are approximately \$122,476 and \$85,615, respectively. Notice the effect of the mortality rate and that possible annuity payments beyond the age of 95 for the (mortal) 65-year-old policy holder do not affect the current price of the annuity.

Exercise 3.2. *(Hedging with index futures)*

(a) If the day-to-day interest is deterministic and $r_{t-1,t} = r_{0,1}$ for all t, then

$$100/B_0 = e^{r_{0,1}+\cdots+r_{t-1,t}} = e^{365 r_{0,1}},$$

from which $r_{0,1} = -(1/365)\log(B_0/100)$ follows. With $Y_0 = 100/B_0$, the leverage of the futures strategy that corresponds to the quadratic hedge is

$$h = \frac{\operatorname{Cov}((S_T-K)_+, S_T Y_0)}{\operatorname{Var}(S_T Y_0)} = \frac{\operatorname{Cov}((S_T-K)_+, S_T)}{Y_0 \operatorname{Var}(S_T)}.$$

Since $S_T = \exp\{\log 100 + 0.035 + 0.1W\}$, the solution to Exercise 1.4 gives formulas for $\operatorname{Cov}((S_T-K)_+, S_T)$ and $\operatorname{Var}(S_T)$. Inserting numerical values gives $h = 0.3156$. The variance of the hedging error is

$$\operatorname{Var}((S_T-K)_+ - hY_0S_T) = \operatorname{Var}((S_T-K)_+)\Big(1 - \operatorname{Cor}((S_T-K)_+, S_T)^2\Big).$$

We can compute the correlation

$$\operatorname{Cor}((S_T-K)_+, S_T) = \frac{\operatorname{Cov}((S_T-K)_+, S_T)}{\sqrt{\operatorname{Var}((S_T-K)_+)\operatorname{Var}(S_T)}}$$

by inserting the expressions obtained in the solution to Exercise 1.4. The computations result in the value 2.933 for the standard deviation of the hedging error.

The quadratic hedge implies a bond position that pays

$$h_0 = \operatorname{E}[(S_T-K)_+] - hY_0 \operatorname{E}[S_T]$$

at time T (in one year). Equivalently, $w_0 = h_0/Y_0$ is the investment in the risk-free bond that corresponds to the quadratic hedge. Again using the expressions from the solution of Exercise 1.4 to compute h_0 and w_0 gives h_0 and $w_0 = -30.9587$.

(b) With h and h_0 from (a), the hedging error is

$$h\frac{100}{97}S_T + h_0 - (S_T - 110)_+, \quad S_T = 100e^{0.035+0.1W},$$

where W is standard normally distributed. Therefore, a sample $\{W_1, \dots, W_n\}$ from $\mathrm{N}(0,1)$ is easily transformed into a sample from the distribution of the hedging error.

(c) Now the time-T value of the long futures strategy with unit leverage is YS_T, where $Y = e^{0.0292+0.05Z}$ and $S_T = 100e^{0.035+0.1W}$ with Z and W being independent and standard normally distributed. Moreover, the money market account is included as a hedging instrument. From the independence of Y and S_T we find that

$$\begin{aligned}
\operatorname{Cov}((S_T-K)_+, YS_T) &= \operatorname{E}[Y]\operatorname{Cov}((S_T-K)_+, S_T),\\
\operatorname{Cov}(YS_T, S_T) &= \operatorname{E}[Y]\operatorname{Var}(S_T),\\
\operatorname{Var}(YS_T) &= \operatorname{Var}(Y)\operatorname{Var}(S_T) + \operatorname{Var}(Y)\operatorname{E}[S_T]^2 + \operatorname{Var}(S_T)\operatorname{E}[Y]^2.
\end{aligned}$$

So the covariance matrix of the hedging instruments (YS_T, S_T) is

$$\Sigma_Z = \begin{pmatrix} \operatorname{Var}(YS_T) & \operatorname{Cov}(YS_T, Y) \\ \operatorname{Cov}(YS_T, Y) & \operatorname{Var}(Y) \end{pmatrix} \approx \begin{pmatrix} 144.46 & 0.2762 \\ 0.2762 & 0.00266 \end{pmatrix}$$

Similarly, the covariances between liability and hedging instruments are

$$\Sigma_{LZ} = \begin{pmatrix} \operatorname{Cov}((S_T-K)_+, YS_T) \\ \operatorname{Cov}((S_T-K)_+, Y) \end{pmatrix} \approx \begin{pmatrix} 36.47 \\ 0 \end{pmatrix}.$$

Using Proposition 3.2 we get the positions in the stochastic hedging instruments:

$$(h_1, h_2) = (0.3152, -32.805),$$

where the first hedging instrument is the futures strategy with unit leverage and the second instrument is on dollar invested in the money market account. The standard deviation of the hedging error is 2.9375 (only slightly higher than before). The position in the zero coupon bond is 0.02 number of bonds (with face value 100).

Exercise 3.3. *(Leverage and margin calls)*

(a) An arbitrage portfolio is obtained by adopting the following strategy:

1. At time 0 take a short position in h forward contracts and a long position in h futures contracts. The net payment is 0.
2. At $t = 1$ you receive $h(F_1 - F_0)$ from the futures contract. Put this into the money market account (with zero interest rate). If it is negative you borrow the same amount. The net cash flow is then again zero.
3. At $t = 2$ the forward contracts gives $h(G_0 - S_2)$, the futures contract $h(S_2 - F_1)$ and the money market account $h(F_1 - F_0)$. Thus the total payoff is
$$h(G_0 - F_0) > 0.$$

(b) If $h(F_1 - F_0) < -K$ you have to borrow at the high interest rate R. Then you need to pay back the interest at $t = 2$, which is $[h(F_0 - F_1) - K](e^R - 1)$. Thus, the total payoff at $t = 2$ is

$$\begin{aligned}
V_2 &= h(G_0 - F_0) - [h(F_0 - F_1) - K](e^R - 1)I\{h(F_1 - F_0) < -K\}\\
&= h(G_0 - F_0) - [h(F_0 - F_1) - K]_+(e^R - 1)
\end{aligned}$$

(c) The expected value of V_2 is given by

$$\operatorname{E}[V_2] = h(G_0 - F_0) - h(e^R - 1)\operatorname{E}[(F_0 - K/h - F_1)_+].$$

The last expectation is identified as the (forward) price of a put option on F_1 with strike $F_0 - K/h$ which can be written as $P_0(F_0 - K/h)$, where

$$P_0(x) = x\Phi(-d_2) - F_0\Phi(-d_1),$$

with

$$d_1 = \frac{\log(F_0/x) + \sigma^2\Delta/2}{\sigma\sqrt{\Delta}}, \qquad d_2 = d_1 - \sigma\sqrt{\Delta}.$$

The maximum expected value is reached for $h = 455$ number of forward contracts.

Exercise 3.4. *(Immunization)*

The discount factors corresponding to the cash flow times $0.5, 1, 1.5$ and 2 years from today are determined as the solution to the equation system (equation (1.3) on page 9)

$$\begin{aligned}
98.51 &= 100d_{0.5}\\
100.71 &= 2d_{0.5} + 102d_1\\
111.55 &= 6d_{0.5} + 6d_1 + 106d_{1.5}\\
198.96 &= 4d_{0.5} + 4d_1 + 4d_{1.5} + 204d_2.
\end{aligned}$$

The solution is $(d_{0.5}, d_1, d_{1.5}, d_2) \approx (0.9851, 0.9680, 0.9418, 0.9185)$. The relation $r_t = -(\log d_t)/t$ transforms the discount factors are transformed into the zero-rates

$$(r_{0.5}, r_1, r_{1.5}, r_2) \approx (0.03002, 0.03248, 0.03997, 0.04249).$$

Other zero-rates are assumed to be given by linear interpolation from those above. In particular, the 20-month rate is

$$r_{5/3} = r_{1.5} + \frac{r_2 - r_{1.5}}{2 - 1.5}(5/3 - 1.5) \approx 0.04081.$$

It follows from Remark 3.1 on page 72 that it is sufficient to use to bonds to make the aggregate position immune against parallel shift in the zero-rate curve, as long as one of the bonds has a duration shorter than that of the liability, $D = 5/3$, and the other has a duration longer than that of the liability. Here,

$$\begin{aligned}
D_C &= \frac{1}{P_C}(0.5 \cdot 6 \cdot d_{0.5} + 1 \cdot 6 \cdot d_1 + 1.5 \cdot 106 \cdot d_{1.5}) \approx 1.42098\\
D_D &= \frac{1}{P_C}(0.5 \cdot 4 \cdot d_{0.5} + 1 \cdot 4 \cdot d_1 + 1.5 \cdot 4 \cdot d_{1.5} + 2 \cdot 204 \cdot d_2) \approx 1.941363.
\end{aligned}$$

A solution to the immunization problem is therefore

$$\begin{pmatrix} h_C \\ h_D \end{pmatrix} = \frac{1}{P_C P_D (D_D - D_C)} \begin{pmatrix} P_D P (D_D - D) \\ P P_C (D - D_C) \end{pmatrix} \approx \begin{pmatrix} 442.0994 \\ 221.6932 \end{pmatrix},$$

where $P = 10^5 e^{-r_{5/3} 5/3} \approx 93424.24$.

Exercise 3.5. *(Delta hedging with futures)*

Chapter 4

Exercise 4.2. *(Sports betting)*

The initial capital is $V_0 = 100$ British pounds and the prices of the contracts paying 1 pound if the gamblers selected result were to happen are $q_1 = 1/2.50$, $q_x = 1/3.25$ and $q_2 = 1/2.70$, respectively. The gambler believes $p_1 = p_x = p_2 = 1/3$.

Buying one contract of each type costs $q_1 + q_x + q_2 = 1.078 > 1$ and pays 1 at maturity whatever happens: a synthetic risk-free bond. Here, this amounts to a guaranteed loss. Anyway, we have a risk-free asset with return $R_0 = 1/(q_1 + q_x + q_2) < 1$. The three risky assets have returns

$$R_1 = q_1^{-1} I\{\text{'Chelsea'}\}, \quad R_x = q_x^{-1} I\{\text{'draw'}\}, \quad R_2 = q_2^{-1} I\{\text{'Liverpool'}\}.$$

However, the risky returns are linearly dependent: one of them can be expressed as a a constant minus a linear combination of the other two. In particular, the covariance matrix of the return vector is not invertible. Therefore we select two of the three risky assets as our risky assets and consider the trade-off problem with the above risk-free asset (the investment problem (4.7) on page 92).

Set

$$\mu' = \begin{pmatrix} \mu'_1 \\ \mu'_2 \end{pmatrix}, \quad \Sigma' = \begin{pmatrix} \Sigma'_{1,1} & \Sigma'_{1,2} \\ \Sigma'_{2,1} & \Sigma'_{2,2} \end{pmatrix},$$

where

$$\begin{aligned} \mu'_1 &= \mathrm{E}[R_x] = p_x/q_x, \quad \mu'_2 = \mathrm{E}[R_2] = p_2/q_2, \\ \Sigma'_{1,1} &= \mathrm{Var}(R_x) = p_x(1-p_x)/q_x^2, \quad \Sigma'_{2,2} = \mathrm{Var}(R_2) = p_2(1-p_2)/q_2^2, \\ \Sigma'_{1,2} &= \mathrm{Cov}(R_x, R_2) = -p_x p_2/(q_x q_2), \quad \Sigma'_{2,1} = \Sigma'_{1,2}. \end{aligned}$$

The solution $\mathbf{w}'$ and w'_0, where $\mathbf{w}' = (w'_1, w'_2)^{\mathrm{T}}$, to the trade-off problem (4.7) with trade-off parameter c is (see (4.8) on page 92)

$$\mathbf{w}' = \frac{V_0}{c}\Sigma'^{-1}(\mu' - R_0\mathbf{1}), \quad w'_0 = V_0 - \mathbf{w}'^{\mathrm{T}}\mathbf{1}.$$

Since we have chosen R_x and R_2 as risky returns, the solution in terms of the capital invested in the outcomes 'Chelsea', 'draw' and 'Liverpool' is

$$\begin{aligned} w_1 &= w'_0 \frac{q_1}{q_1+q_x+q_2}, \\ w_x &= w'_0 \frac{q_x}{q_1+q_x+q_2} + w'_1, \\ w_2 &= w'_0 \frac{q_2}{q_1+q_x+q_2} + w'_2. \end{aligned}$$

The numerical values are

$$\begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix} = \frac{1}{c}\begin{pmatrix} 7.903724 \\ 3.053794 \end{pmatrix}, \quad w'_0 = 100 - \frac{1}{c}10.95752.$$

In particular, for $c = 1$, $(w_1, w_x, w_2) \approx (33.04, 33.32, 33.64)$. As $c \to \infty$ (highly risk-averse), $(w_1, w_x, w_2) \approx (37.10, 28.54, 34.36)$. For $c = 1/10.95752$ (smallest trade-off parameter that corresponds to long positions), $(w_1, w_x, w_2) \approx (0, 72.13, 27.87)$.

The expected value and the variance of the optimal portfolio are

$$\mathrm{E}[V_1] = w'_0 R_0 + \mathbf{w}'^{\mathrm{T}}\mu', \quad \mathrm{Var}(V_1) = \mathbf{w}'^{\mathrm{T}}\mathbf{w}'.$$

Plotting the pairs $(\sqrt{\mathrm{Var}(V_1)}, \mathrm{E}[V_1])$ for $c > 1/10.95752$ illustrates the efficient portfolio frontier.

Exercise 4.3. *(Uncorrelated returns)*

The investment problem and its solution are given by (4.9) and (4.11), respectively, with $R_0 = 1$, $V_0 = 10,000$ and $\sigma_0 V_0 = 30$. The matrix Σ is a diagonal matrix with diagonal entries $\Sigma_{k,k} = \sigma_k^2$. The numerical solution is (approximately)

$$(w_0, w_1, w_2, w_3, w_4, w_5) \approx (8468, 671, 335, 224, 168, 134).$$

Exercise 4.4. *(Hedging a zero-coupon bond)*

(a) The portfolio value in six months is $V_6 = wR + w_0 - L$, where $L = \$10,000$, $w + w_0 \leq V_0 = \$9,700$ and

$$R = \frac{10,000}{9,510}e^{-(\mu+\sigma Z)/4},$$

where $\mu = 0.06$, $\sigma = 0.015$ and Z is standard normally distributed. Since $\mathrm{E}[V_6]$ can be increased by increasing w_0 without increasing $\mathrm{Var}(V_6)$, $w + w_0 = V_0$ for the optimal investment. Therefore, the hedging problem amounts to

$$\begin{aligned}&\text{minimize } w^2 \,\mathrm{Var}(R),\\ &\text{subject to } w\,\mathrm{E}[R] + V_0 - w - L \geq 0.\end{aligned}$$

Since

$$\mathrm{E}[R] = \frac{10,000}{9,510} e^{-\mu/4+(\sigma/4)^2/2} = 1.035877$$

the constraint is equivalent to $w \geq (L - V_0)/(\mathrm{E}[R] - 1) = 8361.943$ and w is chosen as $w = (L - V_0)/(\mathrm{E}[R] - 1) = 8361.943$ in order to minimize the portfolio variance.

(b) For a lognormal random variable e^{a+bZ}, $\mathrm{E}[e^{a+bZ}] = e^{a+b^2/2}$ and $\mathrm{E}[e^{2(a+bZ)}] = e^{2a+2b^2}$ and therefore $\mathrm{Var}(e^{a+bZ}) = e^{2a+b}(e^b - 1)$. Here, this gives $\mathrm{Var}(R) = 1.508974 \cdot 10^{-5}$ and $\sqrt{\mathrm{Var}(R)} = 0.003884552$. The efficient frontier is illustrated by plotting the pairs $(w\sqrt{\mathrm{Var}(R)}, w(\mathrm{E}[R] - 1) + V_0 - L)$ for varying values of w.

Exercise 4.5. *(Hedging stocks with options)*

Exercise 4.6. *(Credit rating migration)*

(a) Let w_1 and $w_2 = V_0 - w_1$, $w_1 \in [0, V_0]$, be the amounts invested in the two bonds, where $V_0 = \$10,000$. The value of the bond portfolio in one year is

$$\begin{aligned}V_1 &= w_1 \frac{100}{83.68} e^{-0.06-s_1+0.012Z} + (10,000 - w_1) \frac{100}{87.50} e^{-0.06-s_2+0.012Z}\\ &= 100 e^{-0.06+0.012Z} \Big(\frac{w_1}{83.68} e^{-s_1} + \frac{10,000 - w_1}{87.50} e^{-s_2} \Big),\end{aligned}$$

where (s_1, s_2) is independent of the standard normal Z and has a distribution specified by Table 4.1. The expected value $\mathrm{E}[V_1]$ and variance $\mathrm{E}[V_1^2] - \mathrm{E}[V_1]^2$ of the portfolio are computed from

$$\mathrm{E}[V_1] = 100 e^{-0.06+0.012^2/2} \sum_{i,j=1}^{4} \Big(\frac{w_1}{83.68} e^{-r_i} + \frac{10,000 - w_1}{87.50} e^{-r_j} \Big) \mathrm{P}((s_1, s_2) = (r_i, r_j))$$

and

$$\mathrm{E}[V_1^2] = 100^2 e^{-0.12+0.024^2/2} \sum_{i,j=1}^{4} \Big(\frac{w_1}{83.68} e^{-r_i} + \frac{10,000 - w_1}{87.50} e^{-r_j} \Big)^2 \mathrm{P}((s_1, s_2) = (r_i, r_j)),$$

where (r_i, r_j) denotes the (i, j) entry in Table 4.1.

Exercise 4.7. *(Insurer's asset allocation)*

We want the solution to a convex optimization problem of the type (2.1): here

$$\begin{aligned}f(\mathbf{w}) &= \frac{1}{2}(\mathbf{w}^{\mathrm{T}} \Sigma \mathbf{w} + \sigma_L^2 - 2\mathbf{w}^{\mathrm{T}} \Sigma_{L,\mathbf{R}}),\\ g_1(\mathbf{w}) - g_{1,0} &= -\mathbf{w}^{\mathrm{T}} \mu + 1.3\,\mathrm{E}[L],\\ g_2(\mathbf{w}) - g_{2,0} &= \mathbf{w}^{\mathrm{T}} \mathbf{1} - 1.2\,\mathrm{E}[L].\end{aligned}$$

The sufficient conditions for an optimal solution in Proposition 2.1 translate into

$$\begin{aligned}
\mathbf{w} &= \Sigma^{-1}(\Sigma_{L,\mathbf{R}} + \lambda_1\mu - \lambda_2\mathbf{1}),\\
\mathbf{1}^{\mathrm{T}}\mathbf{w} &= \mathbf{1}^{\mathrm{T}}\Sigma^{-1}\Sigma_{L,\mathbf{R}} + \lambda_1\mathbf{1}^{\mathrm{T}}\Sigma^{-1}\mu - \lambda_2\mathbf{1}^{\mathrm{T}}\Sigma^{-1}\mathbf{1} = 1.2\,\mathrm{E}[L],\\
\mu^{\mathrm{T}}\mathbf{w} &= \mu^{\mathrm{T}}\Sigma^{-1}\Sigma_{L,\mathbf{R}} + \lambda_1\mu^{\mathrm{T}}\Sigma^{-1}\mu - \lambda_2\mu^{\mathrm{T}}\Sigma^{-1}\mathbf{1} = 1.3\,\mathrm{E}[L]
\end{aligned}$$

so, with $a = \mu^{\mathrm{T}}\Sigma^{-1}\mu$, $b = \mu^{\mathrm{T}}\Sigma^{-1}\mathbf{1}$ and $c = \mathbf{1}^{\mathrm{T}}\Sigma^{-1}\mathbf{1}$,

$$\begin{pmatrix} b & -c \\ a & -b \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1.2\,\mathrm{E}[L] - \mathbf{1}^{\mathrm{T}}\Sigma^{-1}\Sigma_{L,\mathbf{R}} \\ 1.3\,\mathrm{E}[L] - \mu^{\mathrm{T}}\Sigma^{-1}\Sigma_{L,\mathbf{R}} \end{pmatrix}$$

which gives $(\lambda_1,\lambda_2) \approx (232941.2, 242882.4)$. Since the λ_ks are positive we have obtained an optimal solution to the optimisation problem without any constraints on long/short positions. However, the solution is

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \approx \begin{pmatrix} 117647.1 \\ 400000.0 \\ 682352.9 \end{pmatrix}$$

which shows that the solution (luckily) corresponds to an optimal solution to the optimisation problem with a requirement of only long positions.

If the first asset is uncorrelated with the liability, then $\Sigma_{L,\mathbf{R}} = \mathbf{0}$, and again we obtain positive λ_ks, $(\lambda_1,\lambda_2) \approx (231176.5, 240852.9)$, and the solution

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \approx \begin{pmatrix} 47058.8 \\ 400000.0 \\ 752941.2 \end{pmatrix}.$$

In particular, the first asset becomes less attractive when it cannot be used to hedge the liability.

CHAPTER 5

Exercise 5.1. *(Credit Default Swap)*

(a) First observe that buying the defaultable bond and the CDS results in a risk-free payoff of \$100 in 6 months which costs of \$98. Therefore, a rational investor would not invest in the risk-free bond which costs \$99.

Let w_1 and w_2 be the amounts invested in the defaultable bond and in the CDS, respectively. The problem to solve is

$$\begin{aligned}
&\text{maximize } \mathrm{E}\left[u(w_1R_1 + w_2R_2)\right]\\
&\text{subject to } w_1 + w_2 \le 100,\\
&\qquad\qquad w_1, w_2 \ge 0,
\end{aligned}$$

where $R_1 = 100(1-I)/96$ and $R_2 = 100I/2$ with

$$I = \begin{cases} 1 & \text{if the bond issuer defaults,} \\ 0 & \text{otherwise.} \end{cases}$$

Since $u(x) = \sqrt{x}$ and $\mathrm{P}(I=1) = 0.02$,

$$\mathrm{E}\left[u(w_1R_1 + w_2R_2)\right] = \frac{98}{100}\sqrt{w_1\frac{100}{96}} + \frac{2}{100}\sqrt{w_2\frac{100}{2}}.$$

Since $\mathrm{E}\left[u(w_1R_1+w_2R_2)\right]$ is increasing in both w_1 and w_2, $w_1+w_2=100$ for the optimal solution. Therefore the problem to solve simplifies into

$$\begin{aligned}&\text{maximize } \frac{98}{100}\sqrt{w_1\frac{100}{96}}+\frac{2}{100}\sqrt{(100-w_1)\frac{100}{2}}\\&\text{subject to } w_1\in[0,100]\end{aligned}$$

which gives $w_1=\$(98^2\cdot 100)/(2\cdot 96+98^2)\approx \$98.04$ (and $w_2\approx \$1.96$).

(b) *Exercise 5.1 (b) does not make sense. It may read as follows instead: Another investor is an expected-utility maximizer with utility function $u(x)=x^\beta$ for $\beta\in(0,1)$, and invests \$100 in long positions in the defaultable bond and the risk-free bond. Also this investor believes that the default probability is 0.02 and decides to invest less than \$50 dollars in the defaultable bond. What can be said about β?*

Here, let w_1 and w_2 denote the amounts invested in the defaultable bond and in the risk-free bond, respectively. Notice that

$$\mathrm{E}[u(w_1R_1+w_2R_2)]=\frac{98}{100}\Big(w_1\frac{100}{96}+w_2\frac{100}{99}\Big)^\beta+\frac{2}{100}\Big(w_2\frac{100}{99}\Big)^\beta.$$

Since $\mathrm{E}[u(w_1R_1+w_2R_2)]$ is increasing in both w_1 and w_2, $w_1+w_2=100$ for the optimal solution. Therefore the problem to solve simplifies into

$$\begin{aligned}&\text{maximize } \frac{98}{100}\Big(w_1\frac{100}{96}+(100-w_1)\frac{100}{99}\Big)^\beta+\frac{2}{100}\Big((100-w_1)\frac{100}{99}\Big)^\beta\\&\text{subject to } w_1\in[0,100]\end{aligned}$$

Denoting the above objective function by $g(w_1)$, setting $g'(w_1)=0$ and solving for w_1 yield

$$w_1^*(\beta)=100\frac{1-(49/32)^{1/(\beta-1)}}{1+(49/32)^{1/(\beta-1)}/32}.$$

Observe that $w_1^*(\beta)$ is increasing in $\beta\in(0,1)$ with $\lim_{\beta\to 1}w_1^*(\beta)=100$ and $\lim_{\beta\to 0}w_1^*(\beta)=17/49$. Setting $w_1^*(\beta)=50$ and solving for β gives

$$\beta^*=1+\frac{\log(49/32)}{\log(32/65)}=0.398739.$$

We conclude that the parameter β of the investor's utility function satisfies $\beta<\beta^*$.

Exercise 5.2. *(Bets on credit rating)*

Introduce indicator variables X_1,X_2,X_3,X_4 with $X_1=1$ if the rating is 'Excellent' in 6 months and zero otherwise, and similarly for X_2, X_3 and X_4 if the rating is 'Good', 'Poor' or 'Default', respectively. We have four contracts with current and future values given by

$$\begin{aligned}S_0^1&=1,150 & S_6^1&=10,000X_1,\\S_0^2&=8100 & S_6^2&=10,000X_2,\\S_0^3&=700 & S_6^3&=10,000X_3,\\S_0^4&=50 & S_6^4&=10,000X_4.\end{aligned}$$

Let $q_k=S_0^k/10,000$ denote the reciprocal odds of outcome k. The optimization problem to solve can be formulated as follows.

$$\begin{aligned}&\text{maximize } \mathrm{E}[u(w_1q_1^{-1}X_1+w_2q_2^{-1}X_2+w_3q_3^{-1}X_3+w_4q_4^{-1}X_4)]\\&\text{subject to } w_1+w_2+w_3+w_4\le 10,000,\\&\qquad\qquad w_1,w_2,w_3,w_4\ge 0,\end{aligned}$$

where $u(x) = (\gamma x)^{1-1/\gamma}/(\gamma - 1)$ with $\gamma = 2.5$. We identify the investment problem as the "Horse race problem" (5.12) on page 139 and therefore the solution is given by (5.13) on page 140, i.e.

$$w_k = 10,000 \frac{q_k (p_k/q_k)^{2.5}}{\sum_{j=1}^4 q_j (p_j/q_j)^{2.5}}$$

with $p_1 = 0.11$, $p_2 = 0.80$, $p_3 = 0.08$, $p_4 = 0.01$. Inserting the numerical values of the p_ks and q_ks gives (in dollars)

$$w_1 \approx 100.1, \quad w_2 \approx 763.8, \quad w_3 \approx 108.6, \quad w_4 \approx 27.5.$$

Exercise 5.3. *(Hedging with electricity futures)*

Exercise 5.4. *(Optimal payoff function)*

The optimal payoff is given by, see (5.16),

$$h(x) = (u')^{-1}\Big(\lambda \frac{q(x)}{p(x)}\Big),$$

where λ is such that

$$\frac{V_0}{B_0} = \int (u')^{-1}\Big(\lambda \frac{q(x)}{p(x)}\Big) q(x)dx.$$

Here $(u')^{-1}(y) = (y^{-\gamma} - \tau)/\gamma$. Since $q(x)/p(x) = \exp\{\Lambda(\theta) - \theta x\}$, λ must satisfy the equation

$$\begin{aligned}
\frac{V_0}{B_0} &= \int \gamma^{-1}\big(\lambda^{-\gamma} e^{-\gamma(\Lambda(\theta)-\theta x)} - \tau\big) q(x)dx \\
&= \frac{1}{\gamma}\Big[\lambda^{-\gamma} e^{-\gamma\Lambda(\theta)} \int e^{\gamma\theta x} q(x)dx - \tau\Big] \\
&= \frac{1}{\gamma}\Big[\lambda^{-\gamma} e^{-\gamma\Lambda(\theta)+\Lambda(\theta\gamma)} - \tau\Big].
\end{aligned}$$

Therefore,

$$\lambda^{-\gamma} = \Big[\frac{V_0}{B_0}\gamma + \tau\Big] e^{\gamma\Lambda(\theta)-\Lambda(\theta\gamma)}$$

and the resulting optimal payoff is

$$\begin{aligned}
h(x) &= \frac{1}{\gamma}\Big[\Big(\frac{V_0}{B_0}\gamma + \tau\Big) e^{\gamma\Lambda(\theta)-\Lambda(\theta\gamma)-\gamma\Lambda(\theta)+\gamma\theta x} - \tau\Big] \\
&= \frac{1}{\gamma}\Big[\Big(\frac{V_0}{B_0}\gamma + \tau\Big) e^{\gamma\theta x-\Lambda(\theta\gamma)} - \tau\Big].
\end{aligned}$$

CHAPTER 6

Exercise 6.1. *(Convexity and subadditivity)*

Subadditivity and positive homogeneity imply that, for any $\lambda \in [0,1]$,

$$\rho(\lambda X_1 + (1-\lambda)X_2) \leq \rho(\lambda X_1) + \rho((1-\lambda)X_2) = \lambda\rho(X_1) + (1-\lambda)\rho(X_2).$$

Positive homogeneity and convexity imply that

$$\rho(X_1 + X_2) = \rho\Big(2\Big(\frac{1}{2}X_1 + \frac{1}{2}X_2\Big)\Big) = 2\rho\Big(\frac{1}{2}X_1 + \frac{1}{2}X_2\Big) \leq \rho(X_1) + \rho(X_2).$$

Exercise 6.2. *(Stop-loss reinsurance)*

Since $L = \min(S, F_S^{-1}(0.95)) + p$ we find that $F_L(F_S^{-1}(0.95)+p) = 1$ and $F_L(F_S^{-1}(0.95)+p-\varepsilon) < 0.95$ for $\varepsilon > 0$. In particular,

$$F_L^{-1}(0.99) = \min\{m : F_L(m) \geq 0.99\} = F_S^{-1}(0.95) + p.$$

We conclude that $p = F_S^{-1}(0.99) - F_S^{-1}(0.95)$ gives $F_L^{-1}(0.99) = F_S^{-1}(0.99)$.

Exercise 6.3. *(Quantile bound)*

Exercise 6.4. *(Tail conditional median)*

Exercise 6.5. *(Production planning)*

Exercise 6.6. *(Risky bonds)*

(a) Let I_1 and I_2 be the default indicators for the two issuers. They are assumed to be independent and identically distributed, $I_1 = 1$ with probability p and $I_1 = 0$ with probability $1-p$.

The returns of the two bonds are then given by

$$R_k = \frac{10^5}{P_k}(1-I_k) = \frac{R_0}{1-q}(1-I_k), \quad k = 1,2.$$

The expected value of R_k is

$$\mu = E[R_k] = \frac{R_0}{1-q}(1-E[I_k]) = \frac{R_0}{1-q}(1-p) = 1.051, \quad k = 1,2,$$

and the variance is

$$\sigma^2 = V(R_k) = V\Big(\frac{R_0}{1-q}(1-I_k)\Big) = \frac{R_0^2}{(1-q)^2}p(1-p) = 0.02717.$$

Since the default indicators are independent, so are the returns R_1 and R_2, which implies $\mathrm{Cov}(R_1, R_2) = 0$.

Let $\mu^{\mathrm{T}} = (\mu, \mu)$ be the mean vector of $(R_1, R_2)^{\mathrm{T}}$ and Σ be the covariance matrix given by

$$\Sigma = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}.$$

Let w_0 be the amount invested in the risk-free bond and $\mathbf{w}^{\mathrm{T}} = (w_1, w_2)$ the amounts invested in the two defaultable bonds, respectively. The objective is to solve

$$\begin{array}{ll} \text{maximize} & w_0 R_0 + \mathbf{w}^{\mathrm{T}}\mu, \\ \text{subject to} & \mathbf{w}^{\mathrm{T}}\Sigma\mathbf{w} \leq V_0^2\sigma_0^2, \\ & w_0 + \mathbf{w}^{\mathrm{T}}\mathbf{1} \leq V_0, \\ & w_0 \geq 0, w_1 \geq 0, w_2 \geq 0. \end{array}$$

Here $V_0 = 10^6$ is the initial capital and $V_0\sigma_0 = 25000$.

If we, for now, ignore the short-selling constraints, then the sufficient conditions for optimality are

(1) $R_0 - \lambda_2 = 0$ and $-\mu + \lambda_1\Sigma\mathbf{w} + \lambda_2\mathbf{1} = 0$,
(2) $\mathbf{w}^{\mathrm{T}}\Sigma\mathbf{w} \leq V_0^2\sigma_0^2$ and $w_0 + \mathbf{w}^{\mathrm{T}}\mathbf{1} \leq V_0$
(3) $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$,
(4) $\lambda_1(\mathbf{w}^{\mathrm{T}}\Sigma\mathbf{w} - V_0^2\sigma_0^2) = 0$ and $\lambda_2(w_0 + \mathbf{w}^{\mathrm{T}}\mathbf{1} - V_0) = 0$.

Assuming $\lambda_1 > 0$ and $\lambda_2 > 0$ leads to $\lambda_2 = R_0$,

$$\mathbf{w} = \frac{1}{\lambda_1}\Sigma^{-1}(\mu - R_0\mathbf{1}),$$

by (1) and using the first condition in (4) gives

$$V_0^2\sigma_0^2 = \frac{1}{\lambda_1^2}(\mu - R_0\mathbf{1})^{\mathrm{T}}\Sigma^{-1}(\mu - R_0\mathbf{1}).$$

Then we solve for λ_1 which gives

$$\lambda_1 = \frac{1}{V_0\sigma_0}\Big((\mu - R_0\mathbf{1})^{\mathrm{T}}\Sigma^{-1}(\mu - R_0\mathbf{1})\Big)^{1/2}$$

and

$$\begin{aligned}\mathbf{w} &= \frac{V_0\sigma_0}{\Big((\mu - R_0\mathbf{1})^{\mathrm{T}}\Sigma^{-1}(\mu - R_0\mathbf{1})\Big)^{1/2}}\Sigma^{-1}(\mu - R_0\mathbf{1}),\\ w_0 &= V_0 - w_1 - w_2.\end{aligned}$$

We can compute

$$\Sigma^{-1} = \begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & 1/\sigma^2 \end{pmatrix},$$

and putting in the numerical values gives

$$w_1 = w_2 = 107253.1, \qquad w_0 = 785493.8.$$

Since the solution to the optimization problem without short-selling constraints actually satisfies the short-selling constraints we conclude that this is the optimal solution to the problem.

(b) The mean and standard deviation of the optimal portfolio are given by

$$\begin{aligned} w_0R_0 + (w_1 + w_2)\mu &= 1050231,\\ \sqrt{\mathbf{w}^{\mathrm{T}}\Sigma\mathbf{w}} &= 25000.\end{aligned}$$

(c) Let $X = V_1 - V_0R_0$ be the net worth. Then the discounted loss is

$$\begin{aligned} L &= -X/R_0\\ &= -\frac{1}{R_0}\Big(w_0R_0 + w_1(R_1 + R_2) - 1000000R_0\Big)\\ &= (10^6 - w_0) - \frac{w_1}{R_0}(R_1 + R_2)\\ &= (10^6 - w_0) - \frac{w_1R_0}{R_0(1-q)}((1 - I_1) + (1 - I_2))\\ &= \Big(10^6 - w_0 - \frac{2w_1}{1-q}\Big) + \frac{w_1}{1-q}(I_1 + I_2).\end{aligned}$$

The distribution of $I_1 + I_2$ is given by

$$\begin{aligned} P(I_1 + I_2 = 0) &= (1-p)^2,\\ P(I_1 + I_2 = 1) &= 2p(1-p),\\ P(I_1 + I_2 = 2) &= p^2,\end{aligned}$$

and the quantile function is therefore given by

$$F_{I_1+I_2}^{-1}(1-u) = \begin{cases} 0, & \text{if } 1-u \le (1-p)^2, \\ 1, & \text{if } (1-p)^2 < 1-u \le 1-p^2, \\ 2, & \text{if } 1-p^2 < 1-u. \end{cases}$$

The Value-at-Risk is then given by

$$\begin{aligned} \mathrm{VaR}_u(X) &= F_L^{-1}(1-u) \\ &= \Big(10^6 - w_0 - \frac{2w_1}{1-q}\Big) + \frac{w_1}{1-q} F_{I_1+I_2}^{-1}(1-u) \\ &= \Big(10^6 - w_0 - \frac{2w_1}{1-q}\Big) + \frac{w_1}{1-q} \cdot \begin{cases} 0, & \text{if } 1-(1-p)^2 \le u, \\ 1, & \text{if } p^2 \le u < 1-(1-p)^2, \\ 2, & \text{if } u < p^2. \end{cases} \end{aligned}$$

With $u = 0.05$ and $p = 0.024$ we have $1-(1-p)^2 = 0.04742 < 0.05$ and $p^2 = 0.000576$ and therefore

$$\mathrm{VaR}_{0.05}(X) = \Big(10^6 - w_0 - \frac{2w_1}{1-q}\Big) = -5500$$

The Expected Shortfall can be computed as

$$\begin{aligned} \mathrm{ES}_{0.05}(X) &= \frac{1}{0.05}\int_0^{0.05} \mathrm{VaR}_u(X)du \\ &= \Big(10^6 - w_0 - \frac{2w_1}{1-q}\Big) + \frac{2w_1}{0.05(1-q)}\Big(2(p^2-0) + 1(1-(1-p)^2 - p^2)\Big) \\ &= 100103. \end{aligned}$$

(d) Let $I_1 = 1$ with probability $p_1 = 0.91$ and I_2 unchanged. Then

$$\begin{aligned} P(I_1+I_2 = 0) &= (1-p_1)(1-p) = 0.08784, \\ P(I_1+I_2 = 1) &= p_1(1-p) + (1-p_1)p = 0.89032, \\ P(I_1+I_2 = 2) &= p_1 p = 0.02184, \end{aligned}$$

and

$$F_{I_1+I_2}^{-1}(1-u) = \begin{cases} 0, & \text{if } 1-u \le (1-p_1)(1-p), \\ 1, & \text{if } (1-p_1)(1-p) < 1-u \le 1-p_1 p, \\ 2, & \text{if } 1-p_1 p < 1-u. \end{cases}$$

The Value-at-Risk is then given by

$$\begin{aligned} \mathrm{VaR}_u(X) &= F_L^{-1}(1-u) \\ &= \Big(10^6 - w_0 - \frac{2w_1}{1-q}\Big) + \frac{w_1}{1-q} F_{I_1+I_2}^{-1}(1-u) \\ &= \Big(10^6 - w_0 - \frac{2w_1}{1-q}\Big) + \frac{w_1}{1-q} \cdot \begin{cases} 0, & \text{if } 1-(1-p_1)(1-p) \le u, \\ 1, & \text{if } p_1 p \le u < (1-p_1)(1-p), \\ 2, & \text{if } u < p_1 p. \end{cases} \end{aligned}$$

In particular, with $u = 0.05$,

$$\mathrm{VaR}_{0.05}(X) = \Big(10^6 - w_0 - \frac{2w_1}{1-q}\Big) + \frac{w_1}{1-q} = 104503.$$

The Expected Shortfall is given by

$$\begin{aligned}\mathrm{ES}_{0.05}(X) &= \frac{1}{0.05}\int_0^{0.05} \mathrm{VaR}_u(X)du \\ &= 10^6 - w0 - 2*w1/(1-q) + \frac{w_1}{0.05(1-q)}\Big(2(p_1p-0)+1(0.05-p_1p)\Big) \\ &= 152552\end{aligned}$$

Exercise 6.7. *(Leverage and margin calls)*

(a) Recall from the solution to Exercise 3.3 that

$$\begin{aligned}V_2 &= h(G_0-F_0) - [h(F_0-F_1)-K](e^R-1)I\{h(F_1-F_0)<-K\} \\ &= h(G_0-F_0) - [h(F_0-F_1)-K]_+(e^R-1),\end{aligned}$$

where

$$F_1 = F_0\exp\Big\{-\frac{\sigma^2}{2}\Delta + \sigma\sqrt{\Delta}Z\Big\}$$

with Z standard normal, $F_0 = 99.95$, $G_0 = 100$, $\sigma = 0.6$, $\Delta = 1/12$, $R = 0.24$, $K = 10^4$ and $h = 455$. We find that $\mathrm{P}(h(F_0-F_1)-K>0) \approx 0.089$. For $p \le \mathrm{P}(h(F_0-F_1)-K>0)$,

$$\begin{aligned}\mathrm{VaR}_p(V_2) &= F^{-1}_{-V_2}(1-p) \\ &= (e^R-1)F^{-1}_{h(F_0-F_1)-K}(1-p) - h(G_0-F_0) \\ &= (e^R-1)(hF^{-1}_{F_0-F_1}(1-p)-K) - h(G_0-F_0) \\ &= (e^R-1)(h(F_0+F^{-1}_{-F_1}(1-p))-K) - h(G_0-F_0) \\ &= (e^R-1)(h(F_0+F^{-1}_{F_1}(p))-K) - h(G_0-F_0),\end{aligned}$$

where

$$F^{-1}_{F_1}(p) = F_0\exp\Big\{-\frac{\sigma^2}{2}\Delta + \sigma\sqrt{\Delta}\Phi^{-1}(p)\Big\}.$$

Summing up, we have found that, for $p \le 0.05$,

$$\mathrm{VaR}_p(V_2) = a_0 + a_1F^{-1}_{F_1}(p),$$

where

$$a_0 = (e^R-1)(hF_0-K) - h(G_0-F_0), \quad a_1 = (e^R-1)h.$$

(b) Since $\mathrm{ES}_p(V_2) = p^{-1}\int_0^p \mathrm{VaR}_u(V_2)du$ and $F^{-1}_{F_1}(p) = \exp\{m+s\Phi^{-1}(p)\}$ with $m = \log(F_0) - \sigma^2\Delta/2$ and $s = \sigma\sqrt{\Delta}$, Example 6.15 yields, for $p \le 0.05$,

$$\begin{aligned}\mathrm{ES}_p(V_2) &= a_0 + a_1\Phi(\Phi^{-1}(p)-s)e^{m+s^2/2} \\ &= a_0 + a_1F_0\Phi(\Phi^{-1}(p)-\sigma\sqrt{\Delta})\end{aligned}$$

with coefficients a_0 and a_1 as in part (a).

Exercise 6.8. *(Risk and diversification)*

See Figure 1.

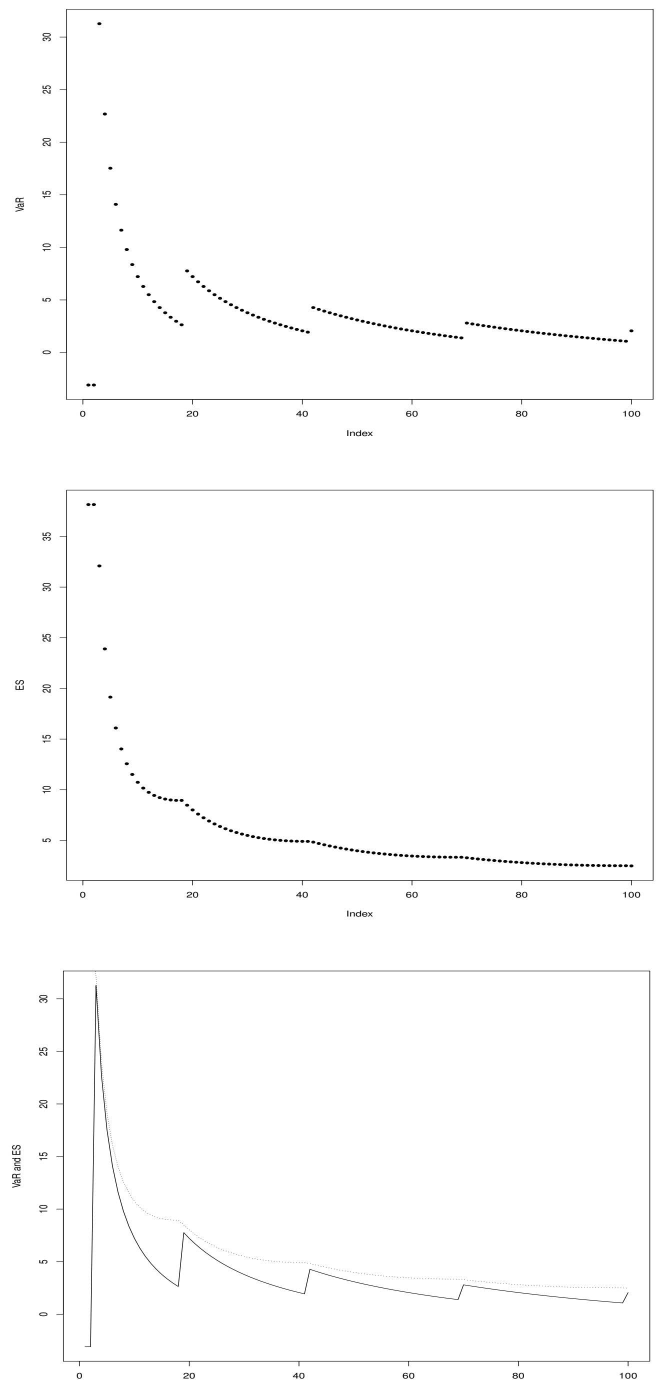

FIGURE 1. Plots of $n \mapsto \mathrm{VaR}_{0.05}(V_1(n) - V_0)$ and $n \mapsto \mathrm{ES}_{0.05}(V_1(n) - V_0)$.