EXAMINATION IN SF2942
PORTFOLIO THEORY AND RISK MANAGEMENT
2013-10-25, 14:00-19:00

The exam consist of 5 questions, with maximum 4 points for each question:
Already passed homework can replace question 1).
Passed Project gives you maximum 3 credit extra (to be added).

Credit scale:

A= at least 18 points, B= at least 16 points,
C= at least 13 points, D= at least 11 points,
E= at least 9 points, Fx= at least 8 points.

Total possible points are 23 = 20 (p. from Exam) + 3 (p. from project).

Examiner: Henrik Shahgholian, with assistance of Sadna Sajadini.

Allowed technical aids: CALCULATOR.

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1) Explain how the bootstrapping procedure works and illustrate it by deriving the zero rates (more precisely, the continuously compounded yearly zero rates) for the maturity times $8, 20, 32, 44$ months, using the bonds listed in the table below. The bonds pay an annual coupon. (*Annual coupon period is counted backwards, i.e.* $T - 12k$.)

Bond	A	B	C
Price	1035573	1055748	1061677
Maturity (months)	8	20	44
Annual coupon (%)	4.25	4.00	3.25
Face value (SEK)	1000000	1000000	1000000

2) An insurance company that has sold unit-linked life insurance contracts pays

$$\min(\max(S_T, K_1), K_2), \qquad 0 < K_1 < K_2,$$

at time T (the end of the year) if the insured dies during the current year. Here S_T is the value of an index and K_1 is a guaranteed amount, and K_2 is the maximum amount paid. Let the random number N of insured that die during the next year be $Po(\lambda)$-distributed (Poisson distribution with mean λ). We assume N is independent of S_T. Denote by L the value of the liability at time T. Suppose there is a liquid market where you can buy and sell call options with strikes K_1 and K_2 and put options with strikes K_1 and K_2. In addition you can buy and sell, at price B_0, a risk free asset that pays 1 at time T. Determine the optimal quadratic hedge of the liability L.

3) Consider an investor who is an expected utility maximizer with a power utility function $u(x) = \sqrt{x}$. The investor has 100 Euro and has the opportunity to take long positions in a defaultable bond and a credit default swap on this bond. One defaultable bond costs 96 Euro now and pays 100 Euro six months from now if the issuer does not default and 0 in case the issuer defaults. The credit default swap costs 2 Euro and pays 100 Euro six months from now if the bond issuer defaults and nothing otherwise.The investor believes that the default probability is 0.02. How much of the 100 Euro does the investor invest in the defaultable bond and in the credit default swap, respectively?

Turn page

4a) Present two quadratic investment principles. Explain the mathematical model assumptions and explain the principles in economic terms.

4b) Define the concept of Duration of a deterministic cash flow for a bond portfolio. Explain it in all details.

4c) Define and explain the Sharpe ratio.

5) Consider an insurance contract that pays the fixed amount 1 at the end of the year in case of a legitimate accident, and nothing otherwise. Let Y_k be the yearly net result for the kth policy, that is, premium income $p+c$ minus any payment to the policy holder, where the yearly premium $p+c$ is greater than the probability p of the occurrence of the event ($c > 0$). For simplicity, all interest rates are assumed to be zero.

a) Determine $\mathrm{VaR}_\alpha(Y_1)$ as a function of α.

b) Consider a large portfolio of size n of identical policies corresponding to independent events whose occurrences all have the same probability. Use a normal approximation of the distribution of $Y_1+\cdots+Y_n$ to approximate $\mathrm{VaR}_\alpha(Y_1+\cdots+Y_n)$. Give an expression for $\mathrm{VaR}_\alpha(Y_1+\cdots+Y_n)$.

c) Take $p = 0.02$, $c = 0.01$, $\alpha = 0.005$, and note that $\Phi^{-1}(0.995) = 2.58$. Determine n_0 such that $\mathrm{VaR}_\alpha(Y_1 + \cdots + Y_n) \leq 0$, for all $n \geq n_0$.

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SOLUTIONS TO EXAMINATION IN SF2942 PORTFOLIO THEORY AND RISK MANAGEMENT 2013-10-25

1) Let C be the $m \times n$ cash-flow matrix of where each row contains the cash-flow payments of the bonds. m is the number of bonds and n the number of cash-flow times. If p is the price vector, then the vector of discount factors d is a solution to $Cd = p$. The typical situation is that m is smaller than n leading to infinitely many solutions. The bootstrapping procedure is used to find a reasonable solution. The bootstrapping procedure works as follows. Start by obtaining the discount factors for the maturities of the zero coupon bonds. In the example it corresponds to bond A. The cash flow times are $t_1 = 8/12$, $t_2 = 20/12$, $t_3 = 32/12$, $t_4 = 44/12$. Then $1035573 = 1.042510^6 d_1$ which gives $d_1 = 0.9934$ and $r_1 = (1/t_1)\log(d_1) = 0.010$. With d_1 determined we can obtain d_2 from bond B. We get

$$p_B = 0.0410^6 d_1 + 1.0410^6 d_2,$$

which leads to d2 = 0.9769 and r2 = 0.014. Now, only one bond remains, but two unknown discount factors. Then, one assumes d3 is given by linear interpolation between d2 and d4 so

$$d_3 = d_2 + \frac{d_4 - d_2}{t_4 - t_2}(t_3 - t_2).$$

This is inserted into the equation for bond C given by

$$p_C = 0.0325.10^6(d_1 + d_2 + d_3) + 1.032510^6 d_4.$$

Solving for d_4 gives $d_4 = 0.9361$, $r_4 = 0.18$, and then $d_3 = 0.9565$ and $r_3 = 0.0167$.

2) The liability at time T is $L = N\min(\max(S_T, K_1), K_2)$. Now investing in the asset S then the time T value is

$$f(S_T) = E(N\min(\max(S_T, K_1), K_2) \mid S_T) = \lambda\min(\max(S_T, K_1), K_2) =$$

$$\lambda\left[\max(S_T - K_1, 0) - \max(S_T - K_2, 0) + K_1\right].$$

This means that the company has to go λ call with strike K_1 long, λ call with strike K_2 short, and $\lambda K_1/B_0$ in the risk free asset.

3) In the presence of the defaultable bond and the credit default swap the risk-free bond is a redundant asset since the payoff of the risk-free bond is the sum of the payoffs of the defaultable bond and the CDS. Since the price of the risk-free bond is greater than the sum of the prices of the defaultable bond and the CDS, it would be suboptimal to invest in it. The solution to

$$\begin{array}{ll} \text{maximize} & E[(w_1 c_1^{-1} X_1 + w_2 c_2^{-1} X_2)^\beta] \\ \text{subject to} & w_1 + w_2 \leq V_0 \\ & w_1 \geq 0, \quad w_2 \geq 0, \end{array}$$

where $X_k \approx Be(p_k)$ and $X_1 + X_2 = 1$, is

$$w_k = V_0 \left[\left(\frac{c_k^\beta}{\beta p_k} \right)^{1/(\beta-1)} \right] \left[\left(\frac{c_1^\beta}{\beta p_1} \right)^{1/(\beta-1)} + \left(\frac{c_2^\beta}{\beta p_2} \right)^{1/(\beta-1)} \right]^{-1},$$

were $c_1 = 0.96$, $c_2 = 0.02$, $p_1 = 0.98$, $p_2 = 0.02$, $V_0 = 100$, $\beta = 0.5$. This gives $(w_1, w_2) \approx (98, 2)$.

4a) See the textbook, Ch 4. Eq. 4.7, and proposition 4.2, Equations 4.9, and 4.10, Proposition 4.3.

4b) See the textbook page 72.

4c) See the textbook, page 93, and also lecture notes.

5a) Since the interest rate is 0 there is no discounting and (draw a picture!) $\text{VaR}_\alpha(Y_k) = F_{-Y_k}^{-1}(1-\alpha)$.

$$F_{-Y_k}(x) = P(I_k \leq x + p + c) = \begin{cases} 0, & x < -p - c, \\ 1 - p, & x \in [-p-c, 1-p-c), \\ 1, & x \geq 1 - p - c. \end{cases}$$

Since $F_{-Y_k}^{-1}(p) = \min\{x : F_{-Y_k}(x) \geq p\}$ we find that for $\alpha \in (0,1)$,

$$\text{VaR}_\alpha = \begin{cases} 1 - p - c, & \alpha < p, \\ -p - c, & \alpha \geq p. \end{cases}$$

5b) Notice that $\text{E}[Y_k] = c$ and $\text{var}(Y_k) = p(1-p)$. Set $S_n = Y_1 + \cdots + Y_n$. Then

$$S_n = \frac{S_n - nc}{\sqrt{np(1-p)}} \sqrt{np(1-p)} + nc$$

which, by the central limit theorem, is approximately normally distributed with mean nc and variance $np(1-p)$. Let Z be standard normally distributed and note that

$$\text{VaR}_\alpha(S_n) \approx \sqrt{np(1-p)}\text{VaR}_\alpha(Z) - nc = \sqrt{np(1-p)}\Phi^{-1}(1-\alpha) - nc,$$

where Φ is the standard normal distribution function.

5c) Setting the above expression to 0 and solving for n gives

$$n = \frac{p(1-p)}{c^2} \Phi^{-1}(1-\alpha)^2 = 1304.654.$$

So $n \geq n_0 \approx 1305$ implies that $\text{VaR}_{0.005}(S_n) \leq 0$.