Antithetic sampling

1. Let X be a standard Gaussian random variable and consider the problem of estimating the quantile $\tau = \mathbb{P}(X > c)$ for some $c > 0$. This can be cast into the framework of Monte Carlo simulation by writing $\tau = \mathbb{E}(\phi(X))$ with $\phi(x) = \mathbb{1}_{\{x>c\}}$ and letting

$$\tau_{N_1} = \frac{1}{N_1} \sum_{i=1}^{N_1} \phi(X_i).$$

However, since ϕ is a monotone function there seems to be good hope for improving this standard Monte Carlo sampler by means of antithetic sampling. More specifically, one may use the alternative estimator

$$\tilde{\tau}_{N_2} = \frac{1}{2N_2} \sum_{i=1}^{N_2} \left(\phi(X_i) + \phi(-X_i)\right),$$

with $\phi(X_i)$ and $\phi(-X_i)$ playing the roles of antithetic variables.

(a) Check that $\tilde{\tau}_{N_2}$ is indeed an unbiased estimator of τ.

(b) In order to have estimators with comparable variance, let $N_1 = 2N_2$ and investigate the gain of variance implied by $\tilde{\tau}_{N_2}$ by computing $\mathbb{V}(\tau_{N_1})/\mathbb{V}(\tilde{\tau}_{N_2})$ as a function of τ. Conclusion?

2. This is a continuation of Problem 3, Exercise session 1, in which the power curve $P(v)$ is monotonously increasing over the interval $(4, 15)$ and decreases only slightly over $(15, 25)$. Use this for reducing the variance of the estimator in (a) via antithetic sampling. Construct a new 95% confidence interval using the robustified estimator and compare it to the ones obtained in (a) and (b).

Asymptotic properties of importance sampling estimators

3. Assume (somewhat academically) that we perform standard importance sampling of a standard Gaussian distribution f using the *scaled Cauchy distribution* as instrumental distribution, i.e.,

$$g(x) = \frac{\gamma}{\pi(\gamma^2 + x^2)}, \quad x \in \mathbb{R}.$$

where $\gamma > 0$.

(a) Compute the importance weight function ω.

(b) We use the previous setup for estimating $\tau = \mathbb{E}_f(\phi(X))$, where $\phi(x) = |x|^{1/2}$, $x \in \mathbb{R}$. Compute the optimal parameter γ minimizing the asymptotic variance of the corresponding importance sampling estimator.

4. (Continuation of Problem 3.) Now consider the converted situation in which we sample a standard Cauchy distribution (i.e. $\gamma = 1$ in this case) using a standard Gaussian distribution as instrumental distribution.

(a) Compute the importance weight function ω.

(b) Let ϕ be the same as in Problem 1. Is there a CLT in this case?

5. (In this problem you may have use of the results obtained in Problem 4, Exercise session 1.) Consider the problem of estimating $\tau = \mathbb{E}_f(\phi(X))$, where the target density $f(x) = z(x)/c$, $x \in \mathsf{X}$, is known up to z only,

by drawing $(X_i)_{i=1}^N$ from some instrumental distribution g (whose support covers that of f) and using the self-normalized importance sampling estimator

$$\tau_N^{\mathrm{SNIS}} = \sum_{i=1}^N \frac{\omega(X_i)}{\sum_{\ell=1}^N \omega(X_\ell)} \phi(X_i),$$

where $\omega(x) = z(x)/g(x)$, $x \in \{x' \in \mathsf{X} : g(x) > 0\}$. We assume that $\mathbb{E}_g(\omega^2(X)\{\phi^2(X) + 1\}) < \infty$.

(a) Show that $\tau_N^{\mathrm{SNIS}} \xrightarrow{\mathbb{P}} \tau$ as $N \to \infty$ (i.e. the estimator is *consistent*).

(b) Establish the CLT

$$\sqrt{N}\left(\tau_N^{\mathrm{SNIS}} - \tau\right) \xrightarrow{\text{d.}} N\left(0, \sigma_g^2(\omega\{\phi - \tau\})/c^2\right) \quad \text{as } N \to \infty$$

(i.e. the estimator is *asymptotically normal*). (Hint: assume first that $\tau = 0$ and use Slutsky's lemma.)

(c) As mentioned in the lectures, the asymptotic standard deviation $\varsigma(\phi) = \sigma_g(\omega\{\phi - \tau\})/c$ is generally intractable. It may however be estimated on the basis of the generated sample $(X_i)_{i=1}^N$. Show, e.g. under the additional assumption $\mathbb{E}_g(\omega^4(X)\{\phi^4(X) + \phi^2(X) + 1\}) < \infty$ [1], that the estimator

$$\varsigma_N(\phi) = \sqrt{N \frac{\sum_{i=1}^N \{\phi(X_i) - \tau_N^{\mathrm{SNIS}}\}^2 \omega^2(X_i)}{\{\sum_{\ell=1}^N \omega(X_\ell)\}^2}}$$

is consistent for $\varsigma(\phi)$ as $N \to \infty$.

[1] This assumption can be weakened.