RULES FOR MANIPULATION FOR SUMMATION

Sf 2955

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Definition: (1) $\sum_{i=1}^{n} x_i = x_1 + x_2 + \ldots + x_n$.

(2) $\sum_{i=1}^{n} a \cdot x_i = a \sum_{i=1}^{n} x_i$.

Proof: Definition (1) gives $\sum_{i=1}^{n} a \cdot x_i = ax_1 + ax_2 + \ldots + ax_n = a\,(x_1 + x_2 + \ldots + x_n) = a \sum_{i=1}^{n} x_i$.

Example: $x_i = 1$, $i = 1 \ldots, n$

$\sum_{i=1}^{n} a = a \sum_{i=1}^{n} 1 = a\,(1 + 1 + \ldots + 1) = a \cdot n$.

(3) $\sum_{i=1}^{n} (x_i + y_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i$.

Proof: Definition (1) gives $\sum_{i=1}^{n} (x_i + y_i) = (x_1 + y_1) + (x_2 + y_2) + \ldots + (x_n + y_n) = x_1 + x_2 + \ldots + x_n + y_1 + y_2 + \ldots + y_n = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i$.

(4) $\sum_{i=1}^{n} (ax_i + by_i) = a \sum_{i=1}^{n} x_i + b \sum_{i=1}^{n} y_i$

Proof: (3) and (2) yield this.

(5) $\sum_{i=1}^{n} (x_i + y_i)^2 = \sum_{i=1}^{n} x_i^2 + 2 \sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i^2$.

Proof: Use $(x_i + y_i)^2 = x_i^2 + 2x_i y_i + y_i^2$ and (4) as well as (2) with $a = 2$.

Let $\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$. Then it holds

(6) $\sum_{i=1}^{n} (x_i - \overline{x}) = 0$.

Proof: $\sum_{i=1}^{n} (x_i - \overline{x}) = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \overline{x}$ enligt (4). But here we have with $a = \overline{x}$ in (2) that $\sum_{i=1}^{n} \overline{x} = \overline{x} \sum_{i=1}^{n} 1 = \overline{x} \cdot n$ according to the Example in (2). But $\overline{x} \cdot n = \sum_{i=1}^{n} x_i$ and this verifies the claim in (6).

(7) $\sum_{i=1}^{n} (x_i - \overline{x}) \cdot (y_i - \overline{y}) = \sum_{i=1}^{n} (x_i - \overline{x})\, y_i = \sum_{i=1}^{n} x_i\, (y_i - \overline{y})$.

Proof: $(x_i - \overline{x}) \cdot (y_i - \overline{y}) = (x_i - \overline{x}) \cdot y_i - (x_i - \overline{x}) \cdot \overline{y}$. Then we get according to (4) that $\sum_{i=1}^{n} (x_i - \overline{x}) \cdot (y_i - \overline{y}) = \sum_{i=1}^{n} (x_i - \overline{x}) \cdot y_i - \sum_{i=1}^{n} (x_i - \overline{x}) \cdot \overline{y}$. But with $a = \overline{y}$ in (2) we get that $\sum_{i=1}^{n} (x_i - \overline{x}) \cdot \overline{y} = \overline{y} \cdot \sum_{i=1}^{n} (x_i - \overline{x})$ and then (6) gives that $\sum_{i=1}^{n} (x_i - \overline{x}) \cdot (y_i - \overline{y}) = \sum_{i=1}^{n} (x_i - \overline{x})\, y_i$. The second identity is obtained analogously.

(8) $\sum_{i=1}^{n}(x_i-\overline{x})\cdot(y_i-\overline{y})=\sum_{i=1}^{n}x_iy_i-n\overline{xy}$.
Proof: Expand. e.g., $\sum_{i=1}^{n}x_i(y_i-\overline{y})$ in the right hand side of (7), use (2) and the definition of $\overline{x}$.

(9) $\sum_{i=1}^{n}(x_i-\overline{x})^2=\sum_{i=1}^{n}x_i^2-n\overline{x}^2$.
Proof: By (5) we get that $\sum_{i=1}^{n}(x_i-\overline{x})^2=\sum_{i=1}^{n}x_i^2-2\sum_{i=1}^{n}x_i\overline{x}+\sum_{i=1}^{n}\overline{x}^2$. Then (2) gives with $a=\overline{x}$ and the Example in (2) that $\sum_{i=1}^{n}x_i^2-2\sum_{i=1}^{n}x_i\overline{x}+\sum_{i=1}^{n}\overline{x}^2=\sum_{i=1}^{n}x_i^2-2\overline{x}\sum_{i=1}^{n}x_i+n\overline{x}^2$. The definition of $\overline{x}$ gives $\sum_{i=1}^{n}x_i=n\overline{x}$, so that $\sum_{i=1}^{n}x_i^2-2\overline{x}\sum_{i=1}^{n}x_i+n\overline{x}^2=\sum_{i=1}^{n}x_i^2-2\overline{x}\cdot n\overline{x}+n\overline{x}^2=\sum_{i=1}^{n}x_i^2-n\overline{x}^2$.