SF2970: Homework 1 (Due on February 8)

Spring 2017

Q 1. (**Conditional expectation**) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let X and Y be integrable random variables.

a.) Let $\mathcal{H} \subset \mathcal{G}$ be two sub-σ-algebras of $\mathcal{F}$. Prove that

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]\,,$$
$$\mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}] = \mathbb{E}[X|\mathcal{H}]\,.$$

(5points)

b.) Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$ such that X is $\mathcal{G}$-measurable. Prove that $\mathbb{E}[Y|\mathcal{G}] = X$ implies that $\mathbb{E}[Y|X] = X$. (Here $\mathbb{E}[Y|X]$ is a shorthand notation for $\mathbb{E}[Y|\sigma(X)]$.)

(5points)

Q 2. (**Martingale I**) Let (Y_n), $n \in \mathbb{N}$, be a simple symmetric random walk on $\mathbb{Z}$. Fix $b \in \mathbb{R}$. Show that

$$S_n := \mathrm{e}^{bY_n} \left(\frac{2}{\mathrm{e}^b + \mathrm{e}^{-b}} \right)^n , \qquad \forall n \in \mathbb{N}\,,$$

is a martingale with respect the natural filtration, i.e. $\mathbb{E}|S_n| < \infty$ and

$$\mathbb{E}[S_{n+1}|\sigma(Y_1, Y_2, \dots, Y_n)] = S_n\,, \qquad \forall n \in \mathbb{N}\,.$$

(10points)

Q 3. (**Martingale II**) Let b, w and r be natural numbers. An urn initially contains b black and w white balls. At each discrete time step (trial), we take out a ball from the urn, determine its color and then return r balls of the same color to the urn (thus at each step we add $r-1$ balls to the urn). Let R_n be the random variable given by the ratio of the number of black balls and the number of total balls in the urn at step n, $n \geq 1$, with $R_1 = b/(b+w)$. Show that (R_n) is a martingale with respect the natural filtration. (10points)

Q 4. (**Girsanov's theorem**) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

a. Let (B_n), $n \geq 0$, be a one-dimensional discrete Brownian motion under $\mathbb{P}$. Compute

$$\mathbb{E}\left[(B_n + n)^2 \exp(-B_n - n/2)\right]\,.$$

(5points)

b. Let $(B_n^{(1)}, B_n^{(2)})$, $n \geq 0$, be a two-dimensional discrete Brownian motion under $\mathbb{P}$. Compute

$$\mathbb{E}\Big[\big(B_n^{(1)} + B_n^{(2)}\big)^2 \exp(-B_n^{(1)} - n/2)\Big]\,.$$

(5points)

Q 5. (**Multivariate Gaussian distribution**) Let Σ be a real symmetric positive definite $n \times n$ matrix, $n > 1$. The n-dimensional normal distribution $\mathcal{N}(0, \Sigma)$ with covariance Σ is given by the density

$$\nu_\Sigma(\mathrm{d}x) := \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp\left(-\frac{1}{2} x^T \Sigma^{-1} x\right) \mathrm{d}^n x\,, \qquad x \in \mathbb{R}^n\,.$$

Let $(X_1, X_2, \ldots, X_n)$ be random variables with joint distribution ν_Σ. Set

$$\mathcal{G} := \sigma(X_2, X_3, \ldots, X_n)\,.$$

Show that the conditional expectation $\mathbb{E}[X_1|\mathcal{G}]$ is almost surely given by

$$\mathbb{E}[X_1|\mathcal{G}] = -\frac{1}{t_{11}} \sum_{j=2}^{n} t_{1j} X_j\,,$$

where (t_{ij}) denote the matrix elements of the inverse matrix T of Σ, i.e., $T = \Sigma^{-1}$.

Hint: Write T in the form

$$T = \left(\begin{array}{c|c} t_{11} & s^T \\ \hline s & S \end{array}\right)$$

with $s^T = (t_{12}, t_{13}, \ldots, t_{1n})$, etc. Choose $G \in \mathcal{G}$ and note that $G = \{\omega : (X_2, \ldots, X_n)(\omega) \in B\}$, for some Borel set B in $\mathbb{R}^{n-1}$. By explicit Gaussian integration check that

$$\mathbb{E}[X_1 \mathbb{1}_G] = \mathbb{E}\Big[-\frac{1}{t_{11}} \sum_{j=2}^{n} t_{1j} X_j \mathbb{1}_G\Big]\,.$$

(10points)

Good luck!