KTH Mathematics

Homework 1 in SF2971 Martingale theory and stochastic integrals, spring 2018.

Due Monday February 12, 2018. Each student should hand in his or her own solutions.

Students taking SF2970 can skip exercises 3 (b) and 4 and do exercise 7 instead.

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1. Let $\Omega = (a, b, c)$ and give a concrete example of a random variable X and σ-algebras $\mathcal{F}$ and $\mathcal{G}$ on Ω such that

$$E\left[E[X|\mathcal{G}]|\, \mathcal{F}\right] \neq E\left[E[X|\mathcal{F}]|\, \mathcal{G}\right].$$

Note that you also have to specify a probability measure on Ω, and that you should demonstrate that the conditional expectations are not the same by computing them!

2. Let $Y_1, Y_2, \ldots$ be nonnegative i.i.d. random variables with $E[Y_i] = 1$. Is the sequence

$$X_n = \prod_{i=1}^{n} Y_i$$

a martingale w.r.t. $\underline{\mathcal{F}} = \{\mathcal{F}_n\}_{n=1}^{\infty}$, where $\mathcal{F}_n = \sigma\{Y_1, Y_2, \ldots, Y_n\}$?

3. Let $\xi_1, \xi_2, \ldots$ be independent with $E[\xi_i] = 0$ and $Var(\xi_i) = \sigma_i^2 < \infty$, and let

$$S_n = \sum_{i=1}^{n} \xi_i \qquad \text{and} \qquad s_n^2 = \sum_{i=1}^{n} \sigma_i^2.$$

(a) Is the sequence $S_n^2 - s_n^2$ a martingale w.r.t. $\underline{\mathcal{F}} = \{\mathcal{F}_n\}_{n=1}^{\infty}$, where $\mathcal{F}_n = \sigma\{\xi_1, \xi_2, \ldots, \xi_n\}$? (Note that $S_n^2 \neq \sum_{i=1}^{n} \xi_i^2$.)

(b) Now assume that $Var(\xi_i) = \sigma^2 < \infty$, $i = 1, 2, \ldots$ and prove **Wald's second equation** that if T is a stopping time with $E[T] < \infty$ then

$$E\left[S_T^2\right] = \sigma^2 E\left[T\right].$$

4. Give an alternative proof of the fact that if X_n, $n \geq 0$ is an $\underline{\mathcal{F}}$-martingale and T is an $\underline{\mathcal{F}}$-stopping time, then the stopped process $X_{\min\{T,n\}}$ is also an $\underline{\mathcal{F}}$-martingale (Lemma 7.3 in the Lecture Notes), using

Theorem Let X_n, $n \geq 0$ be a martingale and H_n a predictable process ($H_n \in \mathcal{F}_{n-1}$) such that $|H_n| \leq M < \infty$. Then $(H \cdot X)_n$ defined by $(H \cdot X)_0 = 0$ and

$$(H \cdot X)_n = \sum_{i=1}^{n} H_i(X_i - X_{i-1}) \qquad n \geq 1$$

is a martingale.

5. If $(\Omega, \mathcal{F}, P)$ is a probability space and L is a nonnegative random variable on $(\Omega, \mathcal{F}, P)$, such that $E[L] = 1$, let

$$Q(A) = \int_A L(\omega) dP(\omega), \qquad \text{for any set } A \in \mathcal{F}.$$

Show that Q is a probability measure on $(\Omega, \mathcal{F})$.

Hint: You may use the following result which follows from the linearity of the integral and the Monotone Convergence Theorem.

Theorem *Let* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of nonnegative measurable functions. Then*

$$\int \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int f_n d\mu.$$

6. Let $\{B_n\}_{n \geq 0}$ be a one-dimensional discrete Brownian motion under P. Compute

(a) $E\left[(B_n + 4n)^2 \exp(-B_n - n/2)\right]$,
(b) $E\left[(B_n + n)^3 \exp(-B_n - n/2)\right]$.

Hint: Use Girsanov's theorem.

7. **Only for students taking SF2970!**

Suppose X_n, $n \geq 1$ is a martingale w.r.t. $\underline{\mathcal{G}} = \{\mathcal{G}_n\}_{n=1}^{\infty}$, and let $\underline{\mathcal{F}} = \{\mathcal{F}_n\}_{n=1}^{\infty}$, where $\mathcal{F}_n = \sigma\{X_1, X_2, \ldots, X_n\}$.

Show that $\mathcal{F}_n \subseteq \mathcal{G}_n$ and that X_n, $n \geq 1$ is a martingale w.r.t. $\underline{\mathcal{F}}$.

Is the same true for any filtration $\underline{\mathcal{H}} = \{\mathcal{H}_n\}_{n=1}^{\infty}$, such that $\mathcal{H}_n \subseteq \mathcal{G}_n$, that is, is X_n, $n \geq 1$ a martingale w.r.t. $\underline{\mathcal{H}}$? Give a proof or a counter example.

Good luck!