KTH Mathematics

Homework 2 in SF2971 Martingale theory and stochastic integrals, spring 2018.

Due Thursday March 1, 2018. Each student should hand in his or her own solutions.

Note: In all stochastic integrals, the integrands can be assumed to be integrable enough to guarantee that the stochastic integral is a martingale.

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1. Solve the stochastic differential equation

$$dX_t = 1 \cdot dt + 2\sqrt{X_t} dB_t, \qquad X_0 = x_0 > 0.$$

where B is a one-dimensional standard Brownian motion. .... (1p)

Hint: Make a transformation of the form $Y_t = u(X_t)$ and look for a **really** simple linear SDE for Y, which allows you to solve for Y and determine the function u.

2. (a) Consider the n-dimensional linear stochastic differential equation

$$\begin{cases} dX_t &= [A(t)X_t + a(t)]\, dt + \sigma(t)dB_t, \quad 0 \le t < \infty, \\ X_0 &= \xi, \end{cases} \tag{1}$$

where $B \in BM(\mathbb{R}^m)$ and independent of the n-dimensional initial vector ξ, and the $n \times n$, $n \times 1$, and $n \times m$ matrices $A(t)$, $a(t)$, and $\sigma(t)$ are assumed to be deterministic, "nice" functions of time.

Let Φ be the unique solution to the (deterministic) matrix differential equation

$$\dot{\Phi}(t) = A(t)\Phi(t), \qquad \Phi(0) = I \tag{2}$$

for $0 \le t < \infty$, where I denotes the $n \times n$ identity matrix. Note that Φ is always non-singular.

Show that

$$X_t = \Phi(t)\left[X_0 + \int_0^t \Phi^{-1}(s)a(s)ds + \int_0^t \Phi^{-1}(s)\sigma(s)dB_s\right]; \quad 0 \le t < \infty,$$

solves (1). .... (0.25p)

(b) Use the result from (a) to solve the 2-dimensional SDE

$$\begin{aligned} dX_t^1 &= X_t^2 dt + \alpha dB_t^1 \\ dX_t^2 &= X_t^1 dt + \beta dB_t^2 \end{aligned}$$

where (B_t^1, B_t^2) is a 2-dimensional Brownian motion, α and β are constants, and $(X_0^1, X_0^2) = (x_0^1, x_0^2)$. .... (0.75p)

Hint: For a constant matrix A the solution to (2) is given by

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k}{k!} t^k.$$

Looking at some known Taylor expansions might also be helpful.

3. Use a stochastic representation result in order to solve the following boundary value problem in the domain $[0, T] \times \mathbb{R}$.

$$\begin{cases} \dfrac{\partial F}{\partial t} + \mu \dfrac{\partial F}{\partial x} + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 F}{\partial x^2} &= 0, \\ F(T, x) &= x^2. \end{cases}$$

Here μ and σ are assumed to be known constants. .... (1p)

4. Let B be a one-dimensional standard Brownian motion on $(\Omega, \mathcal{F}, P^0, \{\mathcal{F}_t\}_{t \geq 0})$, where the filtration is the one generated by B. Fix a time interval $[0, T]$. Define the process X as the solution to the SDE

$$\begin{aligned} dX_t &= \sigma dB_t, \\ X_0 &= 0, \end{aligned}$$

where $\sigma > 0$ is a constant.

(a) Let f be a known real-valued function. Define, for each real number α, a measure P^α, such that X under P^α solves the equation

$$dX_t = \alpha f(X_t) dt + \sigma dB_t^\alpha,$$

where B^α is a Brownian motion under P^α. Give an explicit expression for the Radon-Nikodym derivative (likelihood process)

$$L^\alpha(t) = \frac{dP_t^\alpha}{dP_t^0},$$

where $P_t^\alpha = P^\alpha|_{\mathcal{F}_t}$ and $t \leq T$. .... (1p)

(b) Determine, for every $0 < t \leq T$, the maximum likelihood estimator $\hat{\alpha}(t)$ for the parameter α, based on observations of X over the interval $[0, t]$. In other words: for each fixed t (and ω), solve the problem

$$\max_{\alpha} \quad L^{\alpha}(t),$$

and denote the optimal α by $\hat{\alpha}(t)$.

Now consider two special cases:

i. $f(x) = x$. For this case it is possible to obtain a more explicit expression for $\hat{\alpha}(t)$.

ii. $f(x) \equiv 1$. For this case you should try to express $\hat{\alpha}(t)$ in terms of the observed process X (rather than in terms of the driving Brownian motion).

.... (1p)

(c) For the special case $f(x) \equiv 1$, suppose that $\sigma = 0.1$ and give a 95% confidence interval for the parameter α based on the observations in the time period $[0, t]$. If you want the interval to be reasonably tight (depends on the application of course), say $\hat{\alpha}(t) \pm 0.02$, for how long must you observe X? .... (1p)

Good luck!