KTH Mathematics

Homework 2 in SF2971 Martingale theory and stochastic integrals, spring 2018.

Answers and suggestions for solutions.

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1. The Itô formula applied to $Y_t = u(X_t)$ yields

$$\begin{aligned} dY_t &= u'(X_t)dX_t + \frac{1}{2}u''(X_t)(dX_t)^2 \\ &= u'(X_t)dt + u'(X_t)2\sqrt{X_t}dB_t + \frac{1}{2}u''(X_t)4X_t dt \\ &= \left[u'(X_t) + 2X_t u''(X_t)\right] dt + 2\sqrt{X_t}u'(X_t)dB_t \end{aligned}$$

Now if we want a really simple SDE for Y we could try putting the diffusion equal to one, i.e.

$$2\sqrt{X_t}u'(X_t) = 1.$$

This means that

$$u'(x) = \frac{1}{2\sqrt{x}}$$

and therefore

$$u(x) = \sqrt{x} + C,$$

for some constant C. It will be easiest to put $C = 0$. Now the drift term for Y becomes

$$\frac{1}{2\sqrt{X_t}} + 2X_t\left(-\frac{1}{4X^{3/2}}\right) = 0,$$

so the SDE for Y is

$$dY_t = dB_t, \qquad Y_0 = \sqrt{x_0}.$$

Integrating we obtain

$$Y_t = \sqrt{x_0} + B_t.$$

Finally, we have that

$$X_t = Y_t^2 = (\sqrt{x_0} + B_t)^2.$$

2. (a) Let

$$X_t = \Phi(t)\left[X_0 + \int_0^t \Phi^{-1}(s)a(s)ds + \int_0^t \Phi^{-1}(s)\sigma(s)dB_s\right] = \Phi(t)Y_t, \qquad (1)$$

where

$$dY_t = \Phi^{-1}(t)a(t)dt + \Phi^{-1}(t)\sigma(t)dB_t.$$

Applying the Itô formula to X we find that

$$\begin{aligned} dX_t &= \dot{\Phi(t)}Y_t dt + \Phi(t)dY_t \\ &= A(t)\Phi(t)Y_t dt + \Phi(t)\left(\Phi^{-1}(t)a(t)dt + \Phi^{-1}(t)\sigma(t)dB_t\right) \\ &= \left[A(t)X_t + a(t)\right]dt + \sigma(t)dB_t. \end{aligned}$$

Since $\Phi(0) = I$ we have that

$$X_0 = I \cdot \xi = \xi,$$

and we see that X_t indeed solves the stated SDE.

(b) On matrix form the system looks as follows

$$d\begin{bmatrix} X_t^1 \\ X_t^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} X_t^1 \\ X_t^2 \end{bmatrix}dt + \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} d\begin{bmatrix} B_t^1 \\ B_t^2 \end{bmatrix}$$

From (a) and the hint we see that the solution is given by

$$\begin{bmatrix} X_t^1 \\ X_t^2 \end{bmatrix} = \exp(At)\begin{bmatrix} x_0^1 \\ x_0^2 \end{bmatrix} + \exp(At)\int_0^t \exp(-As)\sigma dB(s)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \quad dB(s) = d\begin{bmatrix} B_s^1 \\ B_s^2 \end{bmatrix}$$

Now noting that $A^2 = I$, where I denotes the 2×2 identity matrix we see that

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k}{k!}t^k = \begin{bmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{bmatrix}$$

and therefore

$$e^{-At} = \sum_{k=0}^{\infty} \frac{(-A)^k}{k!}t^k = \begin{bmatrix} \cosh(t) & -\sinh(t) \\ -\sinh(t) & \cosh(t) \end{bmatrix}$$

and the solution becomes

$$\begin{aligned} dX_t^1 &= x_0^1\cosh(t) + x_0^2\sinh(t) + \alpha\textstyle\int_0^t \cosh(t-s)dB^1(s) + \beta\int_0^t \sinh(t-s)dB^2(s) \\ dX_t^2 &= x_0^1\sinh(t) + x_0^2\cosh(t) + \alpha\textstyle\int_0^t \sinh(t-s)dB^1(s) + \beta\int_0^t \cosh(t-s)dB^2(s) \end{aligned}$$

3. Using a stochastic representation formula tells us that $F(t,x) = E_{t,x}[X_T^2]$ where the dynamics of X are given by

$$\begin{aligned} dX_s &= \mu ds + \sigma X_s dB_s, \\ X_t &= x. \end{aligned}$$

The Itô formula applied to $Y_s = X_s^2$ yields

$$\begin{aligned} d(X_s^2) &= 2X_s dX_s + \frac{1}{2}\cdot 2(dX_s)^2, \\ &= 2X_s\mu ds + 2X_s\sigma X_s dB_s + \sigma^2 X_s^2 ds \\ &= (2\mu X_s + \sigma^2 X_s^2)ds + 2\sigma X_s^2 dB_s. \end{aligned}$$

Integrate to obtain

$$X_u^2 - X_t^2 = \int_t^u (2\mu X_s + \sigma^2 X_s^2)ds + \int_t^u 2\sigma X_s^2 dB_s.$$

Now take expectations given that $X_t = x$ (and assume that it is OK to interchange the order of integration) to arrive at

$$E_{t,x}[X_u^2] = x^2 + \int_t^u \left(2\mu E_{t,x}[X_s] + \sigma^2 E_{t,x}[X_s^2]\right) ds + E_{t,x}\left[\int_t^u 2\sigma X_s^2 dB_s\right]. \quad (2)$$

Assuming enough integrability the expectation of the stochastic integral will be zero. We see that we need the conditional expectation for X_s in order to proceed, but from the SDE for X_s we obtain

$$X_s - X_t = \int_t^s \mu du + \int_t^s \sigma X_u dB_u.$$

Taking expectations given that $X_t = x$ we have that (assuming enough integrability)

$$E_{t,x}[X_s] = x + \mu(s-t) + 0.$$

Inserting this into (2) together with the definition $m(u) = E_{t,x}[X_u^2]$ we have

$$m(u) = x^2 + \int_t^u [2\mu(x + \mu(s-t)] + \sigma^2 m(s))ds + 0.$$

Taking derivatives with respect to time u, and using that $m(t) = x^2$, we arrive at the ODE

$$\begin{cases} \dot{m}(u) &= 2\mu[(x + \mu(u-t)] + \sigma^2 m(u), \\ m(t) &= x^2. \end{cases}$$

This is a first order linear non-homogeneous ODE with the explicit solution

$$m(u) = \left(x^2 + \frac{2\mu x}{\sigma^2} + \frac{2\mu^2}{\sigma^4}\right)e^{\sigma^2(u-t)} - \frac{2\mu^2}{\sigma^2}(u-t) - \frac{1}{\sigma^2}\left(2\mu x + \frac{2\mu^2}{\sigma^2}\right).$$

(The homogeneous solution is $m_h(u) = Ce^{\sigma^2 u}$ for some constant C, and for the particular solution you may try $m_p(u) = Au + B$ for some constants A and B). Finally, we therefore obtain that

$$F(t,x) = m(T) = \left(x^2 + \frac{2\mu x}{\sigma^2} + \frac{2\mu^2}{\sigma^4}\right)e^{\sigma^2(T-t)} - \frac{2\mu^2}{\sigma^2}(T-t) - \frac{1}{\sigma^2}\left(2\mu x + \frac{2\mu^2}{\sigma^2}\right).$$

Note that you can check that you have obtained the correct solution by simply inserting your function into the PDE!

4. Defining the likelihood process by

$$dL^{\alpha}(t) = L^{\alpha} h_{\alpha}(t) dB(t),$$

Girsanov gives us, under P^{α}

$$dB(t) = h_{\alpha}(t)dt + dB^{\alpha}(t)$$

where B^{α} is a P^{α}-Brownian motion.

(a) Substituting into the original SDE we get, under P^{α},

$$dX = \sigma h_{\alpha} dt + \sigma dB^{\alpha}.$$

Thus we must choose h_{α} such that $\sigma h_{\alpha}(t) = \alpha f(X_t)$ i.e.

$$h_{\alpha} = \frac{\alpha}{\sigma} f(X_t).$$

Thus the log-likelihood process is given by

$$\ln L^{\alpha}(t) = \int_0^t \frac{\alpha}{\sigma} f(X_s) dB_s - \frac{1}{2} \int_0^t \frac{\alpha^2}{\sigma^2} f^2(X_s) ds.$$

(b) Maximizing the likelihood is of course equivalent to maximizing the log-likelihood, thus

$$\max_{\alpha} \left\{ \int_0^t \frac{\alpha}{\sigma} f(X_s) dB_s - \frac{1}{2} \int_0^t \frac{\alpha^2}{\sigma^2} f^2(X_s) ds \right\}.$$

Since this is concave in α we obtain the maximum in a point where the derivative w.r.t. α is equal to zero. The optimal α is

$$\hat{\alpha}_t = \frac{\int_0^t f(X_s) \sigma dB_s}{\int_0^t f^2(X_s) ds} = \frac{\int_0^t f(X_s) dX_s}{\int_0^t f^2(X_s) ds}.$$

i. For $f(x) = x$ we obtain

$$\begin{aligned} \hat{\alpha}_t &= \frac{\int_0^t X_s \sigma dB_s}{\int_0^t X_s^2 ds} = \frac{\int_0^t \sigma B_s \sigma dB_s}{\int_0^t X_s^2 ds} == \frac{\sigma^2 (B_t^2 - t)}{2 \int_0^t X_s^2 ds} \\ &= \frac{X_t^2 - \sigma^2 t}{2 \int_0^t X_s^2 ds}. \end{aligned}$$

Note that you prefer the estimate to be expressed in terms of X, since X can be observed, while B can not.

ii. For $f(x) = 1$ we obtain

$$\hat{\alpha}_t = \frac{\int_0^t \sigma dB_s}{\int_0^t ds} = \frac{\sigma B_t}{t} = \frac{X_t}{t}.$$

(c) If α is the true parameter value then our point estimate at time t

$$\hat{\alpha}_t = \frac{X_t}{t} = \frac{\alpha t + \sigma B_t^{\alpha}}{t}$$

is normally distributed with expectation α and standard deviation $\sigma/\sqrt{t}$. Standard theory thus gives us the following 95% confidence interval

$$I_\alpha = \left[\hat{\alpha}_t \pm 1.96\frac{\sigma}{\sqrt{t}}\right].$$

To obtain an interval of the stated length you would nedd to observe X for the time

$$t = \left(\frac{1.96 \cdot 0.1}{0.02}\right)^2 = 96.04.$$

If the unit of time is years you would need a lot of data!