# SF2972 GAME THEORY

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Lecture 1: Jörgen Weibull

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# 1 What is game theory?

A mathematically formalized theory of strategic interaction between

— countries at war and peace, in federations and international negotiations

— political candidates and parties competing for power

— firms in markets, owners and managers, employers and trade-unions

— members of communities with a common pool of resources

— family members and generations who care about each other's well-being

— animals within the same species, from different species, plants, cells

— agents in networks: computers, cell phones, vehicles in traffic systems

## 2 A brief history of game theory

- Emile Borel (1920s): Small zero-sum games

- John von Neumann (1928): the Maxmin Theorem

- von Neumann and Oskar Morgenstern (1944): *Games and Economic Behavior*

- John Nash (1950): Non-cooperative equilibrium ["A Beautiful Mind"]

- John Harsanyi (1960s): Incomplete information

- John Maynard Smith (1970s): Evolutionary stability

# 3 Two interpretations of game theory

John Nash's Ph D thesis, Department of Mathematics, Princeton, 1950.

## 3.1 The rationalistic interpretation

1. The players interact exactly once
2. The players are fully *rational*
3. Each player *knows the game* in question and *knows that all players are rational*

## 3.2 The mass-action interpretation

1. For each player role in the game: a large population of identical individuals

2. The game is recurrently played, each time by randomly drawn individuals, one from each player population

3. Individuals need not be rational, need not know the game, but individuals learn from own and others' experience (in the same player populatioon) to avoid poorly performing strategies

## 3.3 Nash equilibrium

**Definition**: A *Nash equilibrium* is a strategy profile such that if you expect the others to play according to it, then you cannot increase your own payoff by unilaterally changing your strategy.

**Q:** In the rationalistic and/or mass-action interpretation: will players necessarily play a Nash equilibrium?

## 3.4 Examples

A coordination game:

| | $A$ | $B$ |
|---|---|---|
| $A$ | $2, 2$ | $0, 0$ |
| $B$ | $0, 0$ | $1, 1$ |

The matching-pennies game:

| | $H$ | $T$ |
|---|---|---|
| $H$ | $1, -1$ | $-1, 1$ |
| $T$ | $-1, 1$ | $1, -1$ |

| | $L$ | $C$ | $R$ |
|---|---|---|---|
| $T$ | $7, 0$ | $2, 5$ | $0, 7$ |
| $M$ | $5, 2$ | $3, 3$ | $5, 2$ |
| $B$ | $0, 7$ | $2, 5$ | $7, 0$ |

# 4 Key concepts in game theory

1. strategy

2. normal (or strategic) form, extensive form

3. payoff function

4. Nash equilibrium

5. dominance

6. rationalizability

7. evolutionary stability
8. subgame-perfection
9. sequential equilibrium
10. perfect Bayesian equilibrium

# 5 This course

**Purpose**: to introduce key concepts, tools and results

**Main themes**: classical game theory and combinatorial game theory

**Rough outline:**

1. Introduction (today)
2. Normal-form (strategic-form) analysis (Jörgen)
3. Extensive-form analysis (Mark)
4. Combinatorial game theory (Jonas)

## 5.1 Readings

1. "A Course in Game Theory" by Martin Osborne and Ariel Rubinstein (available electronically on AR's home page)

2. "On Numbers and Games", by John Conway (two almost identical editions)

3. "Winning Ways, Volumes 1-4" (available electronically at KTHB. Vol 1 fun to have).

## 5.2 Examination

Take-home exercises and a written exam

# 6 Simple applications of game theory

## 6.1 Natural resources

- Two fishermen, fishing in the same area

- Each fisherman can either fish modestly, $M$, or aggressively, $A$. The profits are

$$\begin{array}{ccc} & M & A \\ M & 3,3 & 1,4 \\ A & 4,1 & 2,2 \end{array}$$

- Both prefer $(M, M)$, and both dislike $(A, A)$

- If each of them strives to maximize his or her profit, and they are both rational: $(A, A)$

- Unregulated competition leads to over-exploitation

- Adam Smith's "invisible hand" fails

- This game is called a *Prisoners' Dilemma*, where $M$ is to *cooperate*, $D$ to *defect*

## 6.2 Market competition

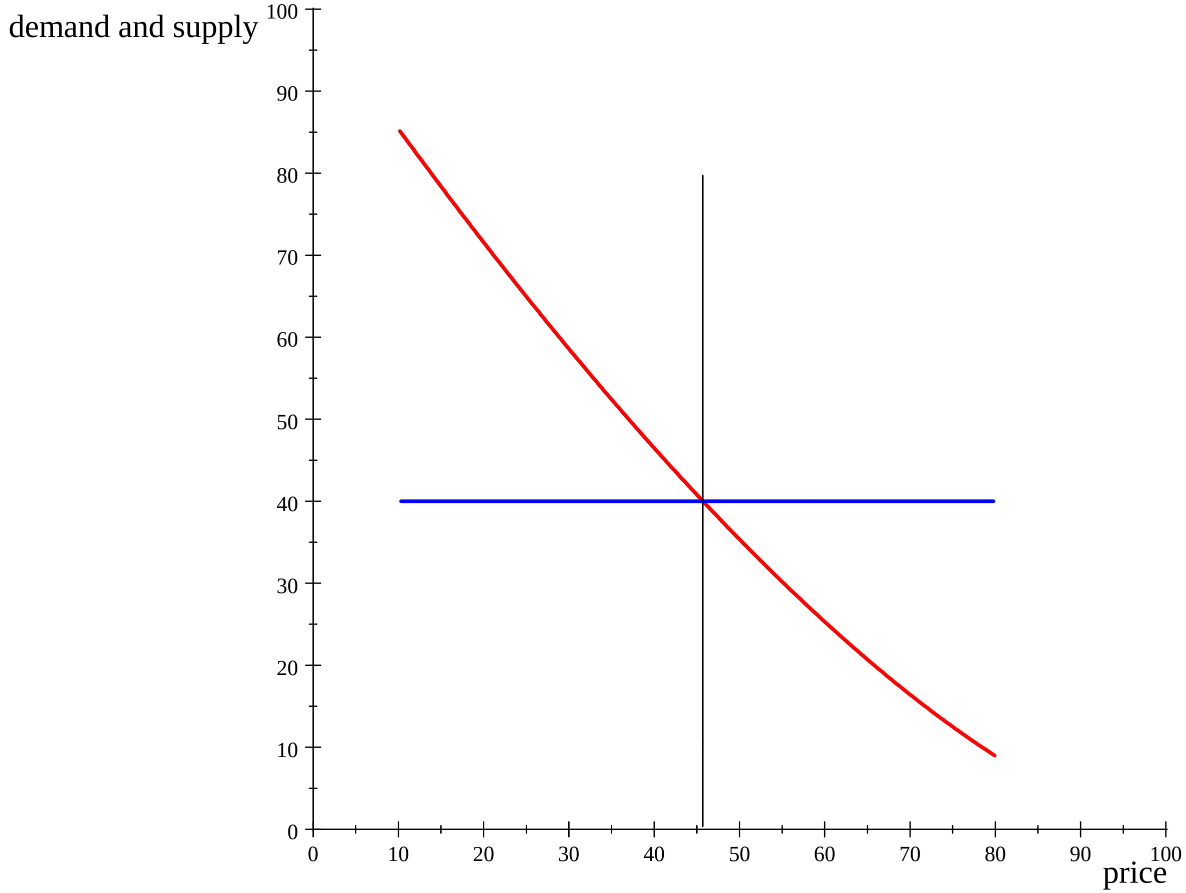
- $n$ firms competing in a homogeneous product market

- All firms simultaneously select output levels $q_1, q_2, ..., q_n$

- Let $Q = q_1 + ...q_n$

- The market clears at a price $p$ such that

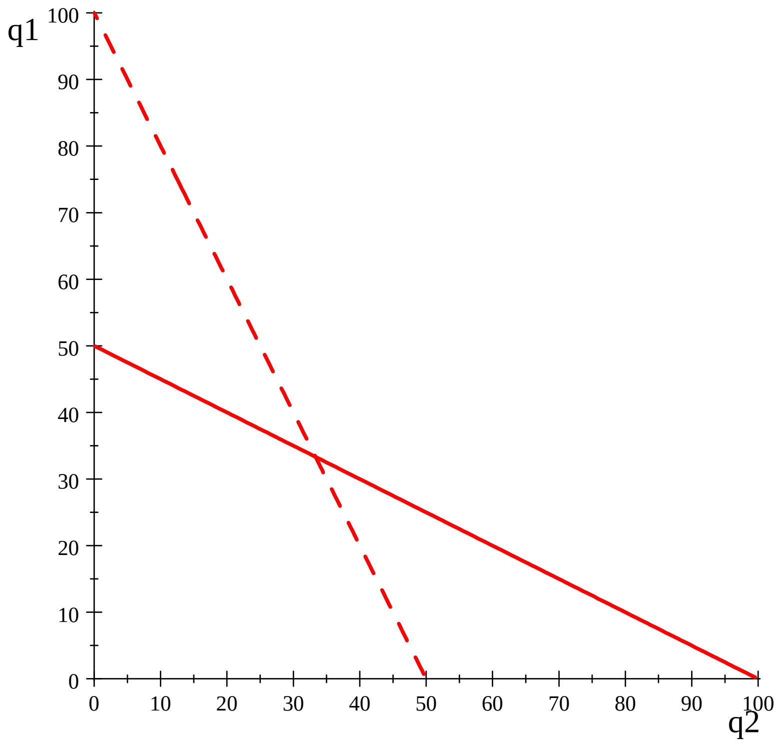
$$D(p) = Q$$


demand and supply
100
90
80
70
60
50
40
30
20
10
0
0
10
20
30
40
50
60
70
80
90
100
price

- Suppose *you are the manager of firm* $i$, that all firms have the same production costs, and the managers of all firms are rational profit-maximers who know that the others are rational too.

- What level of output, $q_i$, would you choose?

- Solution in the case of duopoly, $n = 2$:




- Cournot 1839

## 6.3 Partnerships

- Business partners, or pair of students writing an essay at KTH

- Each partner has to choose between W ("work") and S ("shirk")

  - If both choose W: expected *gain to both*

  - If one chooses W and the other S: *loss* to the first and *gain* to the second

  - If both choose S: expected *heavy loss to both*

$$\begin{array}{ccc} & W & S \\ W & 3,3 & -1,4 \\ S & 4,-1 & -2,-2 \end{array}$$

- This is **not** a Prisoners' Dilemma: S does *not dominate* D: it is (after all) better to "work" if the other "shirks"

- If you were one of the two individuals, what would expect from the other, what would you do?

- In a large population, with random matching, would $(W, W)$ be a stable outcome? Would $(S, S)$?

# 7 Mathematics

## 7.1 Notation and tools

1. **Useful sets**: $\mathbb{N}$ the positive integers, $\mathbb{R}$ the reals, $\mathbb{R}_+$ the non-negative reals

2. **Definition**: *Upper-contour sets* for a function $f : \mathbb{R}^n \to \mathbb{R}$

$$\{x \in \mathbb{R}^n : f(x) \geq \alpha\}$$

3. **Definition**: *Convex* sets $X \subset \mathbb{R}^n$

$$x, y \in X \Rightarrow \lambda x + (1 - \lambda) y \in X \quad \forall \lambda \in [0, 1]$$

4. **Definition**: A function $f : X \to \mathbb{R}$ with convex domain $X$, is *quasi-concave* if all its upper contour-sets are convex

5. **Notation**: Given a function $f : X \to \mathbb{R}$:

$$\arg\max_{x \in X} f(x) = \{x^* \in X : f(x^*) \geq f(x) \;\; \forall x \in X\}$$

6. **Definition:** A *correspondence* $\varphi$ from a set $X$ to a set $Y$, written

$$\varphi : X \rightrightarrows Y$$

is a function that assigns a *non-empty set* $\varphi(x) \subset Y$ to each $x \in X$

## 7.2 Theorems

**Definition:** A set $X \subset \mathbb{R}^n$ is *compact* if it is closed and bounded.

**Theorem 7.1 (Weierstrass' Maxium Theorem)** *If $X \subset \mathbb{R}^n$ is non-empty and compact, and $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, then also* $\arg\max_{x \in X} f(x)$ *is non-empty and compact.*

**Definition**: For any set $X$ and function $f : X \to X$, $x \in X$ is a *fixed point* if $f(x) = x$.

**Theorem 7.2 (Brouwer's Fixed-Point Theorem)** *If $X \subset \mathbb{R}^n$ is non-empty, compact and convex, and $f : X \to X$ is continuous, then there exists at least one fixed point.*