SF2972 GAME THEORY
Lecture 2

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January 19, 2011

A game theorist's approach to applications

1. Identify key aspects of the strategic interaction in question

2. Simplify as much as possible, without losing what seem to be the most essential features

3. Write up an extensive-form game that represents the interaction

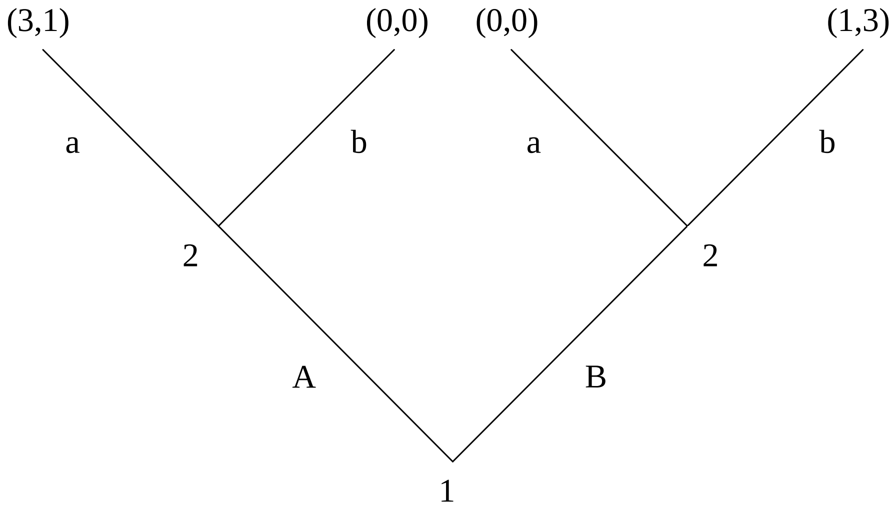
4. Write up a normal-form representation of the extensive-form game

5. Analyze the extensive-form game (hard) or analyze the normal-form game (usually easier), or do both.

6. If step 5 is successful, go back to step 2, but simplify less, and do steps 3-5. Terminate when you have interesting enough results for a sufficiently rich model.

1 Informally about the extensive form

- Is it always better to be more informed?



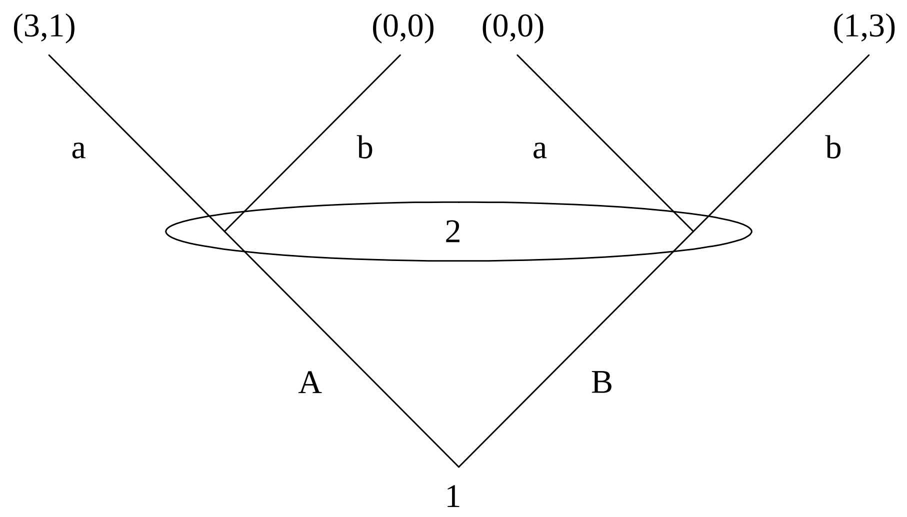
Game 1

- How many pure strategies does each player have?

- *Backward induction*

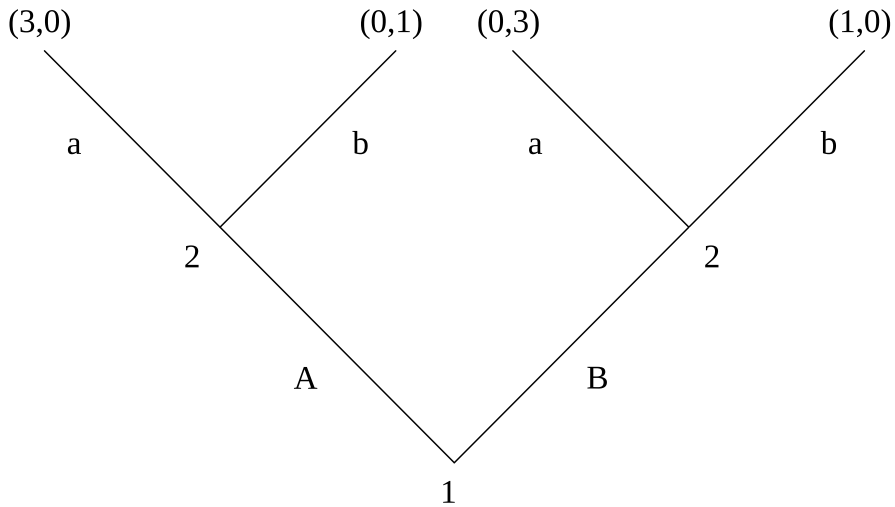
- *Perfect*-information games vs. games of *imperfect* information

- Suppose that player 2 is *not* informed about 1's move:



Game 2

- In this game, player 2 cannot condition his choice on 1's action
- How many pure strategies does each player have?
- In Game 1: *First*-mover advantage (better to be less informed)
- Are there games with a *second*-mover advantage?



Game 3

2 Informally about the normal form

Game 1:

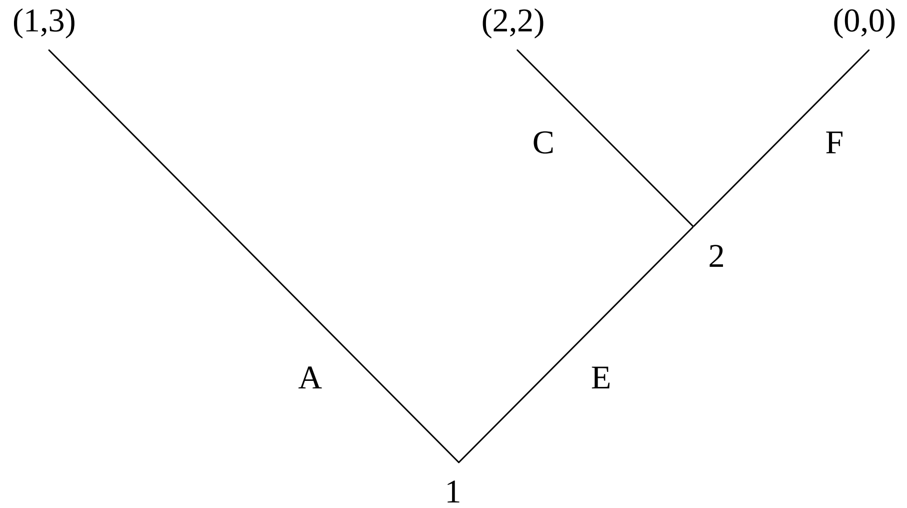
	aa	ab	ba	bb
A	$(3,1)$	$(3,1)$	$(0,0)$	$(0,0)$
B	$(0,0)$	$(1,3)$	$(0,0)$	$(1,3)$

Game 2:

	a	b
A	$(3,1)$	$(0,0)$
B	$(0,0)$	$(1,3)$

3 Extensive forms with the same normal form

An *entry-deterrence* game: A potential entrant (player 1) into a monopolist's (player 2) market

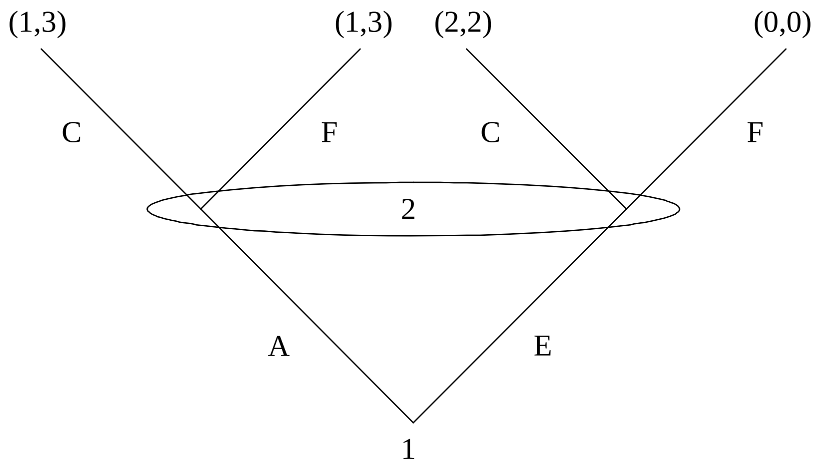


Game 4

- Its normal form:

	C	F
A	$1,3$	$1,3$
E	$2,2$	$0,0$

- Another extensive form game with the same normal form:



Game 5

4 Preferences, utility functions and payoff functions

- A set X of *alternatives* $x, y, z...$

- Preferences as *binary relations* $\succcurlyeq$ on X: $x \succcurlyeq y$

  * *Transitivity*: if $x \succcurlyeq y$ and $y \succcurlyeq z$, then $x \succcurlyeq z$

  * *Completeness*: either $x \succcurlyeq y$ or $y \succcurlyeq x$ or both

- Indifference $x \sim y$ and strict preference $x \succ y$

Let $\succcurlyeq$ be a binary relation on a set X.

Definition 4.1 *A* **utility function** *for $\succcurlyeq$ is a function $u : X \to \mathbb{R}$ such that $u(x) \geq u(y)$ iff $x \succcurlyeq y$.*

5 Decision-making under uncertainty

- Let the alternatives $x \in X$ be risky investment opportunities, gambles, outcomes or plays in a game.

5.1 Expected-utility theory

- John von Neumann and Oskar Morgenstern: *The Theory of Games and Economic Behavior* (1944)

- Let each alternative $x \in X$ be a probability distribution over a finite set T of possible outcomes (or plays) $\tau_1, ..., \tau_m$:

$$X = \Delta(T) = \{x \in \mathbb{R}_+^m : \sum_{k=1}^{m} x_k = 1\}.$$

- Let $\succcurlyeq$ be a player's preferences over such "lotteries" $x \in X$

- Question: Does there $\exists$ a function $v : T \to \mathbb{R}$ such that

$$x \succcurlyeq y \quad \Leftrightarrow \quad \sum_k x_k \cdot v(\tau_k) \geq \sum_k y_k \cdot v(\tau_k) \quad ?$$

- If yes, then

$$u(x) = \sum_{k=1} x_k \cdot v(\tau_k)$$

  is a utility function $u : X \to \mathbb{R}$ for $\succcurlyeq$ on X

- v is called a *Bernoulli function* or *von Neumann-Morgenstern utility function*

- *The existence of such a function v is called the expected-utility hypothesis*

5.2 Payoff functions in game theory

A two-step procedure:

1. For each player, define a *Bernoulli function* over the set of possible plays of the game

2. Given these Bernoulli functions, the *payoff function* for a player maps *strategy profiles* to the player's expected Bernoulli function values

6 Normal-form games

- Normal-form game = Game in strategic form = Strategic game (Osborne-Rubinstein)

Definition 6.1 *A* **normal-form game** *is a triplet* $G = \langle N, S, u \rangle$*, where*

(i) $N = \{1, 2, ..., n\}$ *is the set of* **players**

(ii) $S = \times_{i \in N} S_i$ *is the set of* **strategy profiles** $s = (s_1, ..., s_n)$*, with* S_i *denoting the* **strategy set** *of player* i

(iii) $u : S \to \mathbb{R}^n$ *is the* **combined payoff function***, where, for each strategy profile* $s \in S$ *and player* $i \in N$*,* $u_i(s)$ *is player* i*'s* **payoff (utility)**

- Notation: for any strategy profile $s \in S$, player $i \in N$ and strategy $s_i' \in S_i$, write $\left(s_i', s_{-i}\right)$ for the strategy profile in which s_i has been replaced by s_i'

- Notation: for any strategy profile $s \in S$ and player $i \in N$, write

$$\begin{aligned}\beta_i(s) &= \arg\max_{s_i' \in S_i} u_i\left(s_i', s_{-i}\right) \\ &= \left\{s_i' \in S_i : u_i\left(s_i', s_{-i}\right) \geq u_i\left(s_i'', s_{-i}\right) \quad \forall s_i'' \in S_i\right\}\end{aligned}$$

- This defines player i's *best-reply correspondence* $\beta_i : S \rightrightarrows S_i$

- Write $\beta(s) = \times_{i \in N} \beta_i(s) = \beta_1(s) \times \beta_2(s) \times ... \times \beta_n(s)$

- This defines the *combined best-reply correspondence* $\beta : S \rightrightarrows S$

Definition 6.2 *A strategy profile* $s^* \in S$ *is a Nash equilibrium (NE) if* s^* *is a best reply to itself;* $s^* \in \beta(s^*)$.

6.1 Examples

- Reconsider the finite games in Lecture 1

- The Cournot duopoly: assume that each firm strives to maximize its profit

$$\pi_i\,(q) = (100 - Q)\,q_i$$

6.2 Ordinal games

Definition 6.3 *An* **ordinal normal-form game** *is a triplet* $G = \langle N, S, (\succcurlyeq_i)\rangle$, *where*

(i) $N = \{1, 2, ..., n\}$ *is the set of players*

(ii) $S = \times_{i \in N} S_i$ *is the set of strategy profiles* $s = (s_i)_{i \in N}$ *with* S_i *denoting the strategy set of player* i

(iii) For each $i \in N$*,* $\succcurlyeq_i$ *is player* i*'s* **preference ordering** *of the set* S *of strategy profiles.*

- Interpretation: $s \succcurlyeq_i s'$ means that player i (weakly) prefers strategy profile s over strategy profile s' [or, more exactly, (weakly) prefers the probability distribution over outcomes/plays that is induced by s over that induced by s']

- For any strategy profile $s \in S$ and player $i \in N$, we now write

$$\beta_i(s) = \left\{ s'_i \in S_i : \left(s'_i, s_{-i}\right) \succcurlyeq_i \left(s''_i, s_{-i}\right) \quad \forall s''_i \in S_i \right\}$$

- Nash equilibrium can be defined in the same way as with payoff functions, that is, as a strategy profile that is a best reply to itself.

- Note that, if, for each player $i \in N$, u_i is a utility function for player i, then a strategy profile is a NE in the ordinal game $\langle N, S, (\succcurlyeq_i) \rangle$ iff it is a NE in the game $\langle N, S, u \rangle$.

7 Existence of Nash equilibrium

Recall:

Definition 7.1 *A function* $f : X \to \mathbb{R}$*, where* X *is a convex set, is* **quasi-concave** *if, for each* $a \in \mathbb{R}$*, the upper-contour set*

$$X_a = \{x \in X : f(x) \geq a\}$$

is convex.

Definition 7.2 *A* **fixed point under a correspondence** $\varphi : X \rightrightarrows X$ *is a point* $x \in X$ *such that* $x \in \varphi(x)$*.*

Theorem 7.1 *Let $G = \langle N, S, u \rangle$ be a normal-form game in which each strategy set S_i is a non-empty, compact and convex set in some Euclidean space $\mathbb{R}^{m_i}$, each payoff function $u_i : S \to \mathbb{R}$ is continuous, and quasi-concave in the player's own strategy, $s_i \in S_i$. Then G has at least one Nash equilibrium.*

Proof sketch:

1. Weierstrass' Maximum Theorem $\Rightarrow \beta(s)$ non-empty and compact $\forall s \in S$

2. Quasi-concavity $\Rightarrow \beta(s)$ convex $\forall s \in S$

3. Berge's Maximum Theorem $\Rightarrow \beta$ upper hemi-continuous

4. Kakutani's Fixed-Point Theorem: Every upper hemi-continuous correspondence φ from a non-empty, compact and convex set S to itself has at least one fixed point if $\varphi(s)$ is non-empty, compact and convex $\forall s \in S$

- Nash's (1950) existence result is a special case:

Definition 7.3 *For any player i in any game: a* **mixed strategy** *x_i is a randomization (probability distribution) over the player's strategy set S_i;*

$$x_i \in \boldsymbol{\Delta}(S_i) = \left\{ x_{ih} \in \mathbb{R}_+^{m_i} : \sum_{h \in S_i} x_{ih} = 1 \right\}$$

Theorem 7.2 (Nash, 1950) *Let $G = \langle N, S, u \rangle$ be a normal-form game in which the set N of players is finite, each strategy set S_i is non-empty and finite. Then G has at least one Nash equilibrium in pure or mixed strategies.*

Proof: Each player's set of mixed strategies is a non-empty compact and convex set in a Euclidean space, and each player's expected payoff is a continuous function, that is linear in the player's own mixed strategy. Hence, the general existence theorem, given above, applies.