SF2972 GAME THEORY
Lecture 7

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1 Examples

1.1 Pure price competition [Bertrand duopoly]

1. Two firms with identical products

2. Both firms have the same constant unit cost $c \geq 0$

3. Demand $D(p) = \max\{0, 100 - p\}$

4. Each firm chooses its price $p_i \in [0, 100]$ so as to maximize its profit

$$\pi_i^*(p_1, p_2) = \begin{cases} (p_i - c) D(p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c) D(p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases} \qquad (\text{for } j \neq i)$$

- Formalize this as a normal-form game $G = \langle N, s, u \rangle$

- Do best replies always exist?

- Draw a picture of the graph of π_1

- Identify the set of pure-strategy equilibria

- Discuss weak dominance

- Consider the case of $c = 10$ and a smallest monetary unit for prices:

$$p_i \in \{0, 1, 2, ..., 99, 100\}$$

1.2 Pure location competition [the Downs model]

1. Two firms, same product, but at locations x and y in $[0, 1]$

   - or two firms that choose different varieties of a product

   - or two political candidates/parties, taking policy positions in competition for votes

2. Both firms have the same constant unit cost $c > 0$

3. Both firms sell at the same (fixed) price $p > c$

4. Each firm strives to maximize its profit, or, equivalently, its sales

5. Consumers uniformly distributed on $[0, 1]$ and each consumer buys 1 unit from the nearest seller

- Formalize this as a normal-form game $G = \langle N, s, u \rangle$

- Do best replies always exist?

- Identify the set of pure-strategy Nash equilibria

- Consider the case of a finite and odd set of locations,

$$X = \{x_1, x_2, ..., x_{2m+1}\} \subset [0, 1]$$

  at equal distance from each other, with 1 consumer at each location

- Will best replies now always exist? Nash equilibrium?

1.3 Competition in both location and price [the Hotelling model]

A two-stage game:

- Stage 1: Both firms simultaneously choose their locations, x and y
- Stage 2: Both firms observe each others' locations and simultaneously choose their prices, p_1 and p_2
- Each firm strives to maximize its (expected) profit
- What is now the strategy set of firm 1, of firm 2?
- Non-credible threats and the idea of subgame perfection

1.4 Sequential quantity competition [Stackelberg leadership]

- Reconsider the Cournot duopoly, with two identical firms, constant unit cost $c > 0$, but now interacting in two stages

- Stage 1: Firm 1 chooses its output quantity, q_1

- Stage 2: Firm 2 observes q_1 and chooses its own output quantity, q_2

- Each firms strives to maximize its profit.

- What is the strategy set of firm 1, firm 2?

- Non-credible threats and the idea of subgame perfection

2 Subgame perfection

- The idea goes back to Selten (1965) [Osborne-Rubinstein 6.2]

- The concept will be fully formalized in Mark's lectures on finite extensive-form games

- Here we develop the idea rather informally

Definition 2.1 *A* **subgame** *of an extensive-form game* Γ *is the extensive-form game* Γ_a *that*

(i) has as its initial node ("root") a singleton node a *such that if a node* b *in an information set* I *is preceded by node* a*, then all nodes in* I *are preceded by* a

(ii) "inherits" the Bernoulli function values of Γ*.*

- Note that, by definition, Γ is a subgame of itself (just as any set is a subset of itself).

Definition 2.2 *A strategy profile s in an extensive-form game Γ is* **subgame perfect** *if it induces a Nash equilibrium in all subgames of Γ.*

- Reconsider examples

3 Games of incomplete information

- In many strategic interactions, the actors know the "rules of the game" but not each others' preferences (or payoff functions)

- Such situations of *incomplete* information are usually modelled as games of *imperfect* information [Harsanyi (1967)]

- Approach: Create a "meta-game" by

  (a) Introducing "nature" as "player 0", who makes an initial random draw from a set of possible preference profiles

  (b) Let each player learn his or her preference, or "type"

  (c) Let each player decide what to do, conditional each of his or her possible types

- Formally, this gives rise to an extensive-form game with an initial random move by "nature"

3.1 Example

- Two competing firms with private information about their own production costs

- Suppose that both know the *prior* probabilities, μ, for the four possible cost constellations: (c_L, c_L), (c_L, c_H), (c_H, c_L), (c_H, c_H)

- Each firm learns its own cost ("type") and uses Bayes' law to infer the *posterior* probability distribution for the other firm's cost. For instance, suppose firm 1 learns that its cost is low. Then:

$$\Pr\left[\text{Firm 2's cost is } c_L \mid \text{1's cost is } c_L\right] = \frac{\mu\left(c_L, c_L\right)}{\mu\left(c_L, c_L\right) + \mu\left(c_L, c_H\right)}$$

etc.

- Having done this, they simultaneously take some action (output quantity, price and/or location)

- Each firm takes his or her two actions in order to maximize its expected profit, given its posterior belief about the other firm's cost and action

- Note that this two-player game can alternatively be analyzed as a four-player game, in which each firm has been replaced by two players, one for each of its two “types” (its low- and high-cost incarnations)

- What is the strategy set of each player in each of these two game representations? How define payoff functions?

4 Bayesian games

- A general model of strategic interactions under incomplete information [Osborne-Rubinstein 2.6]

Definition 4.1 *The* **pre-image** *of a set* $B \subset Y$ *under any function* $f : X \to Y$*, is the set*

$$f^{-1}(B) = \{x \in X : f(x) \in B\}$$

1. Let $N = \{1, 2, ..., n\}$ be the set of (personal) players i

2. Let Ω be a finite set of *states of nature* ω

3. For each player i, let T_i be a finite set of potential *signals* or *types*

4. For each player i, let $\tau_i : \Omega \rightarrow T_i$ be i's *signal function* [complete information being the special case $\tau_i(\omega) \equiv \omega$]

5. For each player i, let $\mu_i \in \Delta(\Omega)$ be i's *prior belief*, before i receives her/his signal

   - and assume that each signal has a positive prior:

$$\mu_i \left[\tau_i^{-1}(t_i)\right] > 0 \quad \forall t_i \in T_i$$

6. For each player i, let A_i be a set of *actions* a_i [the strategies in the "underlying" game]

   and write $A = \times_{i \in N} A_i$

7. For each player i, let S_i be the set of functions $s_i : T_i \rightarrow A_i$

   - the *strategies* for player i in the "Bayesian game" specifying what action to take, conditional upon each of i's signal

8. Whenever this game is played, exactly one point in the set $\Omega \times A$ will materialize

9. Preferences $\succsim_i$ are defined over the set $\Delta(\Omega \times A)$ of probability distributions ("lotteries") over $\Omega \times A$

   - if these preferences satisfy the vonNeumann-Morgenstern axioms: let $v_i : \Omega \times A \to \mathbb{R}$ be a Bernoulli function representation of $\succsim_i$

10. This defines an *ordinal Bayesian game* $\left\langle N, \Omega, (A_i, T_i, \tau_i, \mu_i, \succsim_i)_{i \in N} \right\rangle$, or a *Bayesian game* if we replace preferences by Bernoulli functions u_i.

- One may represent this either as an n-player normal-form game. However:

Definition 4.2 *A strategy profile s in a Bayesian game* Γ *is a* **Nash equilibrium** *if implements a Nash equilibrium in the normal-form game* $G^* = \langle N^*, A^*, u^* \rangle$*, where*

(a) $N^* = \{(i, t_i) : i \in N$ *and* $t_i \in T_i\}$

(b) $A^*_{(i,t_i)} = A_i$ *and* $A^* = \times_{(i,t_i) \in N^*} A^*_{(i,t_i)}$

(c) $u^*_{(i,t_i)}(a^*) = \mathbb{E}_{(i,t_i)}[v_i(\omega, a(\omega))]$*, where the expectation is taken with respect to* (i, t_i)*'s posterior,*

$$\mu_{(i,t_i)}(\omega) = \begin{cases} \mu_i(\omega) / \mu_i\left[\tau_i^{-1}(t_i)\right] & \text{if } \omega \in \tau_i^{-1}(t_i) \\ 0 & \text{otherwise} \end{cases}$$

and $a_j(\omega) = a^*_{(j,\tau_j(\omega))}$ *for all* $j \in N$ *and* $\omega \in \Omega$

5 Examples

5.1 Cournot duopoly with uncertain costs

- Two firms competing in a homogeneous product market

- Firm 1 has unit production cost c

- Firm 2 has either unit production cost c_L or c_H where $c_L < c_H$

- The probabilities are $\Pr[c_L] = \lambda$ and $\Pr[c_H] = 1 - \lambda$ for some $\lambda \in (0, 1)$

- Firm 2 learns its own cost, but firm 1 is not informed of 2's actual cost

- Then both firms simultaneously select output levels, q_1 and q_2

- The market clears at the price $p = F(q_1 + q_2)$

- Formalize this as a three-player game and solve for NE

5.2 Second-price auction

- An indivisible object is auctioned off to the highest bidder in a sealed-bid procedure

- The bidder with the highest bid wins the object and pays the second highest bid

  [A so-called Vickrey auction]

- Suppose there are n bidders, and that the bidders' valuations are statistically independent draws from the same probability distribution π on some finite set $V \subset \mathbb{R}$ of potential valuations

- Hence, $\Omega = V^n$ and $\omega = (v_1, ..., v_n)$

- Assume all bidders know this: a common prior $\mu = \pi^n$:

$$\mu(v_1, ..., v_n) = \pi(v_1) \cdot ... \cdot \pi(v_n)$$

- Each bidder is only informed about his or her valuation:

$$\tau_i(v_1, ..., v_n) \equiv v_i$$

- What is a bidder's set of pure strategies?

- How define payoff functions?

- Homework: Show that bidding one's valuation is a (weakly) dominant strategy, and that this strategy profile constitutes a Nash equilibrium. [Osborne and Rubinstein Example 27.1]

Next lecture

1. Interpretations of mixed strategies

2. Rationalizability

3. Evolutionary stability