SF2972 GAME THEORY
Problem Set 3
Due February 24 at the lecture

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1. Consider the price competition between two firms with identical products. Each firm i has a constant unit cost c_i of production, where $0 \leq c_i \leq 20$, and demand is given by $D(p) = \max\{0, 100 - p\}$. Each firm chooses its price $p_i \in [0, 100]$ so as to maximize its profit

$$\pi_i^*(p_1, p_2) = \begin{cases} (p_i - c_i) D(p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c_i) D(p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases} \qquad (\text{for } j \neq i)$$

   (a) Suppose first that one of the firms would be a monopolist (the other firm being absent). What would be the (optimal) monopoly price for firm i in such a monopoly situation?

   (b) Suppose that both firms are in the market and $c_1 = c_2 = c$. Find the unique (pure-strategy) Nash equilibrium price and compare with the monopoly price at that unit cost.

   (c) Is the Nash equilibrium price in (b) undominated?

   (d) Does a (pure-strategy) Nash equilibrium exist if $c_1 \neq c_2$?

   (e) Do (b)-(d) in the case of a smallest monetary unit: $c_i, p_i \in V = \{1, 2, ..., 99, 100\}$ for $i = 1, 2$.

2. Consider two firms that sell the same product, but at locations x and y in some set $X \subset [0, 1]$. Both firms have zero production costs. Consumers are uniformly distributed on $L = [0, 1]$, and each consumer buys 1 unit from the nearest seller (they split the market evenly if $x = y$). Each firm strives to maximize its profit (its market share times its price).

   (a) Suppose first that both firms sell at the same fixed price $p > 0$. Write this up as a normal-form game and find its unique Nash equilibrium, (x^*, y^*), when $X = L$.

   (b) Do (a) when $X \subset [0, 1]$ is an arbitrary set such that $1/2 \in X$.

   (c) Suppose now that each firm can choose its price after they both have chosen locations. More exactly: In stage 1 both firms simultaneously choose their locations, $x \in X$ and $y \in X$. In stage 2, both firms observe each others' locations, x and y, and simultaneously choose their prices, $p_1 \geq 0$ and $p_2 \geq 0$. A consumer located at any point $z \in L = [0, 1]$ buys from firm 1 if

$$p_1 + \tau \cdot |x - z| < p_2 + \tau \cdot |y - z|$$

and from firm 2 if the inequality is reversed. Here $\tau > 0$ is a (transportation cost) parameter for consumers. In case of equality, the consumer buys with equal probability from any one of the two firms. Each firm strives to maximize its (expected) profit.

(b1) What is the strategy set of firm 1, of firm 2, in this two-stage game? Define the firms' payoff functions [their profits as functions of their strategies].

(b2) Let $X = \{1/4, 1/2, 3/4\}$. For what parameter values $\tau > 0$, if any, is it a Nash equilibrium outcome that they locate at $x = 1/4$ and $y = 3/4$ and choose the same price? [Hint: the firms may have to make (non-credible) price-threats against each others' alternative locations.]

(b3) Let $X = \{1/4, 1/2, 3/4\}$. For what parameter values $\tau > 0$, if any, is it a subgame-perfect equilibrium outcome that they locate at $x = 1/4$ and $y = 3/4$ and set the same price? What is the range of subgame perfect equilibrium prices at those locations? [Hint: solve by backward induction, by first considering price competition at all possible location pairs.]

3. Consider two firms competing in a homogeneous product market. Firm 1 has unit production cost $c = 10$. Firm 2 has either unit production cost c_L or c_H where $0 \leq c_L < c_H \leq 20$. The probabilities are $\Pr[c_L] = \lambda$ and $\Pr[c_H] = 1 - \lambda$, where $\lambda \in (0, 1)$. Both firms know the cost of Firm 1, and Firm 2 knows its own cost, but Firm 1 is not informed of 2's cost. Both firms simultaneously select output levels, q_1 and q_2, in $[0, 100]$. The market clears at the price $p = \max\{0, 100 - q_1 - q_2\}$. Formalize this as a three-player game (between Firm 1, Firm 2L and Firm 2H) and solve for Nash equilibrium.

4. An indivisible object is auctioned off to the highest bidder in a sealed-bid procedure. The bidder with the highest bid wins the object and pays the second highest bid. Suppose that there are two bidders, and that the bidders' valuations are statistically independent draws from the uniform distribution on the finite set of potential valuations $V = \{1, 2, ..., 99, 100\}$ (that is, probability $1/100$ for each value). Assume that both bidders know this, so this is their common prior, but each bidder i is only informed about his or her own valuation, v_i.

   (a) Formalize this as a (Bayesian) normal-form game with two players. What is a pure strategy in this game? Is it a Nash equilibrium to always bid one's valuation? Prove or disprove!

   (b) Formalize this as a normal-form game with 200 ($= 2 \cdot 100$) players. What the (pure) strategy set of a player in this game? Is it a Nash equilibrium for each player to bid his or her "type"? Prove or disprove!

   (c) In (a): Is the strategy to always bid one's valuation a weakly dominant strategy, in the sense that, in comparison with any other strategy, it never does worse and sometimes does better than that alternative strategy?

5. Evolutionary stability

(a) Find all (pure or mixed) evolutionarily stable strategies in

	L	C	R
T	$7,0$	$2,5$	$0,7$
M	$5,2$	$3,3$	$5,2$
B	$0,7$	$2,5$	$7,0$

(b) Find all (pure or mixed) evolutionarily stable strategies in

	H	D
H	$-1,-1$	$4,0$
D	$0,4$	$2,2$

(c) Write up the normal form of the extensive-form given below. How many pure strategies does each player have? Verify that it is a symmetric game. Suppose that $v = 5$. Is it an evolutionarily stable strategy to always move left? Is the strategy profile in which both players always move left a subgame-perfect equilibrium? Discuss your findings!