SF 2972 GAME THEORY
Problem set 1

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1. Consider the two-player normal-form game:

$$\begin{bmatrix} 1,1 & a,0 & 0,a \\ 0,a & 1,1 & a,0 \\ a,0 & 0,a & 1,1 \end{bmatrix}$$

   (a) Find all (pure and mixed) Nash equilibria when $a = 0$.

   (b) Find all (pure and mixed) Nash equilibria when $a = 1$.

   (c) Find all (pure and mixed) Nash equilibria when $a = 2$.

   (d) For each Nash equilibrium in (a)-(c), and each player: identify the player's set of pure best replies to the equilibrium in question.

   (e) Find all (pure and mixed) perfect equilibria, when $a = 0$, $a = 1$ and $a = 2$.

2. Consider the two-player game:

	A	B	C	D
A	$4,4$	$8,3$	$0,0$	$0,0$
B	$3,8$	$7,7$	$0,0$	$0,0$
C	$0,0$	$0,0$	$2,2$	$1,1$
D	$0,0$	$0,0$	$1,1$	$1,1$

   (a) Find all pure strategies that are strictly dominated (by a pure or mixed strategy), and for each such strategy (if such exist) specify the (pure or mixed) strategy by which it is dominated.

   (b) Find all rationalizable pure strategies for each player.

   (c) Find all pure-strategy Nash equilibria.

   (d) Find all pure-strategy perfect equilibria.

3. Consider the two-player normal-form game:

	L	R
T	$7,6$	$0,5$
B	$2,0$	$4,3$

(a) Find all Nash equilibria, perfect equilibria, and proper equilibria (in pure and mixed strategies).

(b) Suppose that the players can talk with each other before they play. If you are player 1 (the row player) and player 2 (the column player) says "Let us play (T,L), it gives us the highest possible payoffs!", would that increase the likelihood that you will play T? If both players care only about their own payoffs (as given in the payoff bi-matrix), and not about honesty in communication and other values, does 2's message then contain any information about his intention?

4. Consider two firms in price competition in a market for a homogeneous good. The firms simultaneously set their prices, p_1 and p_2, from some given set $P \subseteq \mathbb{R}_+$. All customers buy from the firm with the lowest price. If both firms set the same price, then each firm receives half of the market demand at that price. Both firms produce at a constant unit cost, c_1 and c_2, respectively, and have no fixed costs. Let market demand be $D(p) \equiv a - p$ for some $a > 0$, and assume that $0 \leq c_1 \leq c_2 < a$.

(a) For each firm, find its monopoly price (the price it would have set, had it been alone in the market).

(b) For $c_1 = c_2$ and $P = [0, a]$: Define the associated normal-form game (strategy sets and payoff functions). Are the payoff functions continuous? Do best replies always exist? Does there exist Nash equilibria in pure strategies? Does there exist any Nash equilibrium in undominated strategies?

(c) For $c_1 < c_2 < a$ and $P = [0, a]$: The same questions as in (b).

(d) Do (b) and (c) for $P = \{0, 1, 2, ..., a\}$ when $0 < c_1 \leq c_2 < a$ are integers.

5. Suppose one individual, Anne, owns an indivisible object. Anne is considering selling this object to a prospective buyer, Bert. The object is worth w to Anne and v to Bert, where $v, w \in [0, 1]$. If the object is sold at a price p, then Anne's utility is $p - w$; she has to give up the object worth w to her but receives p in return. Similarly, Bert's utility from such a transaction is $v - p$. If no trade occurs, then each party obtains utility zero. Suppose that this is a truly bilateral situation; none of the parties can trade with a third party.

   (a) Verify that Pareto efficiency (that is, that there exists no allocation that is better for both) requires that the object be sold if $w < v$ and not if $w > v$, and illustrate this in a diagram for valuation pair (w, v) in the unit square.

   (b) Now suppose that Bert does not know Anne's valuation, so w is Anne's *private information*. Similarly, suppose that v is Bert's private information. To be more precise, suppose that the two individuals have been randomly drawn from a population of individuals with different valuations, in such a way that their "types", w and v, are statistically independent and identically distributed random variables. Suppose that the trading mechanism is that the seller, Anne, makes a take-it-or-leave-it offer to Bert, and suppose that the population value distribution is uniform over $[0, 1]$. What price should Anne then set in order to maximize her profit? Should the price depend on Anne's own valuation?

   (c) Suppose that Anne and Bert have decided to instead use the double auction format, under the same statistical assumption as in the preceding task. In this mechanism, both parties simultaneously write down a price and puts it in a sealed envelope. Then the envelopes are opened and trade takes place at the average of these two prices if the seller's ask price is lower than the buyer's bid price. Otherwise no trade takes place. Verify that the following strategy pair constitutes a Nash equilibrium: Anne's *ask price* is $p_A = 2w/3 + 1/4$, unless her valuation w exceeds $3/4$, in which case she asks her true valuation, $p_A = w$. Likewise, Bert's *bid price* is $p_B = 2v/3 + 1/12$, unless his valuation v falls short of $1/4$, in which case he bids his true valuation, $p_B = v$. Show also that trade occurs if and only if Bert's valuation, v, exceeds Anne's valuation, w, by at least $1/4$, and indicate the *area of no trade* in your diagram in (a).