SF 2972 GAME THEORY
Solutions to PS 1

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February 9, 2013

1. (a) Let $a = 0$, and let us name the three pure strategies A, B, and C. There are 3 pure NE: (A, A), (B, B), and (C, C). The unique pure best reply to (A, A) is A, to (B, B) it is B and to (C, C) it is C. There are 4 mixed NE. All pure strategies in their supports are best replies, and only these. Three of the mixed equilibria involve 50/50 randomizations between pairs of pure strategies: $\{A, B\}$, $\{B, C\}$ and $\{A, C\}$, respectively:

Player 1		Player 2	
probability	pure best reply	probability	pure best reply
A (1/2), B (1/2)	A or B	A (1/2), B (1/2)	A or B
A (1/2), C (1/2)	A or C	A (1/2), C (1/2)	A or C
B (1/2), C (1/2)	B or C	B (1/2), C (1/2)	B or C

   The fourth mixed NE involves a uniform randomization over all three strategies, each played with probability 1/3. All pure strategies are best replies. All in all there thus are 7 NE.

   (b) Let $a = 1$. There are still 3 pure NE: (A, A), (B, B), and (C, C). These are no longer strict; each player has an alternative best reply to each equilibrium. (For example: C is a best reply to (A, A).) There is no mixed NE involving pairs of pure strategies. There is still one mixed NE involving all three strategies, each played with probability 1/3. All pure strategies are best replies to this (completely) mixed NE. All in all there are 4 NE (an even number, a non-generic case).

   (c) Let $a = 2$. There is no pure NE. There is no mixed NE involving pairs of pure strategies. There is still one mixed NE involving all three strategies, each played with probability 1/3. All pure strategies are best replies to this (completely) mixed NE. All in all there is 1 NE.

   (d) Done above.

   (e) For $a = 0, 1, 2$: No pure strategy is (weakly) dominated, so all NE are undominated, and hence perfect equilibria, since this is a 2-player game.

2. (a) Pure strategy B is strictly dominated by a randomization over strategies A and C, where the probability to play A, p_A, satisfies $7/8 < p_A < 1$.

   (b) The rationalizable strategies for each player are A, C, and D.

   (c) The pure-strategy NE are (A, A), (C, C), and (D, D).

   (d) The candidates are the three NE. However, since strategy D is weakly dominated by C, (C, C) is not perfect. The other two NE are perfect.

3. (a) NE: (T, L), (B, R) and the mixed strategy profile x, where $x_1 = (3/4, 1/4)$ and $x_2 = (4/9, 5/9)$. All strict, and also all completely mixed NE are perfect and proper, so all three NE in this example have these robustness properties.

   (b) In class. A reasonable answer is "no" to both questions, since player 1 cannot deduce 2's intention from 2's message (2 would gain from 1 playing T even if 2 plans to play R). In order to analyze this question rigorously, one needs to incorporate the message-sending stage into the strategic analysis. For instance, one can write up a so-called *cheap-talk game*, in which both players first send messages to each other (say, simultaneously), and then play the game in question. A pure strategy for a player is then a message to send and, for each possible message pairs, a strategy to play in the game (that may be conditioned on the messages sent). This is a fascinating part of game theory, with some surprising (mostly negative) results.

4. (a) The monopoly price for each firm i is found by solving the profit-maximization program

$$\max_{p \in [0,a]} \quad (a - p)(p - c_i)$$

   which has the unique solution

$$p_i^{mon} = \frac{a + c_i}{2} \in (0, a)$$

   (b) Let $c = c_1 = c_2$. The normal-form game is $G = (N, S, \pi)$, where $N = \{1, 2\}$, $S_1 = S_2 = P = [0, a]$, $S = P^2$ and, for $i, j \in N$ with $j \neq i$:

$$\pi_i(p_1, p_2) = \begin{cases} (a-p)(p-c) & \text{if } p_i < p_j \\ \frac{1}{2}(a-p)(p-c) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

   These payoff functions are discontinuous and firm i has no best reply to $p_j \in (c, p^{mon})$. Nevertheless, there exists a unique (pure-strategy) NE, namely marginal-cost pricing: $p_1 = p_2 = c$. We note that each player's equilibrium strategy is weakly dominated (for example by setting the monopoly price).

   (c) In this case there exists no NE (in pure strategies), unless firm 2's marginal cost is quite high and the marginal cost difference is large. The reason is that, when the marginal costs are not very high and not very different, the more efficient firm 1 "wants to" price slightly below the marginal cost of firm 2, to "capture" the whole market, but (with continuum strategy sets) there exists no optimal such under-cutting price, unless firm 1's monopoly price is lower than firm 2's marginal cost. In the latter case, that is, when $p_1^{mon} \leq c_2$, or, equivalently, $a \leq 2c_2 - c_1$, there exists infinitely many NE, all of the form $p_1 = p_1^{mon}$, $p_2 > p_1$, and all resulting in firm 1 "capturing" the whole market. (Consider, for example, the case when $a = 1$, $c_1 = 0.1$ and $c_2 = 0.6$.)

(d) In this case we have a finite game. Hence, in the mixed-strategy extension, there always exists at least one Nash equilibrium and, indeed at least one perfect equilibrium, and hence at least one Nash equilibrium in undominated strategies. But we here focus on pure strategies. In the case $c_1 = c_2 = c$ there now are two pure NE: $p_1 = p_2 = c$ and $p_1 = p_2 = c + 1$, respectively, where the former NE is weakly dominated while the second is undominated (and hence perfect, since this is a two-player game). In the case $c_1 < c_2$: *either* $a \leq 2c_2 - c_1$, in which case the Nash equilibria in (c) are still Nash equilibria, *or* $a > 2c_2 - c_1$, in which case $p_1 = c_2 - 1$ and $p_2 = c_2$ constitutes a NE. This is undominated iff $c_1 < c_2 - 1$.

5. (a) Pareto efficiency means that it is not possible to make one party strictly better off without making the other party worse off. Hence, Pareto efficiency requires that the object be allocated to the party who values it most. Pareto efficiency thus requires that the good be sold at all valuation pairs above the diagonal in the unit square in (w, v)-space.

(b) Since there are no point masses in the type distributions, ties can be neglected (these have probability zero). Bert will accept any offer $p < v$. Anne thus chooses her ask price p so as to maximize

$$u_A(p) = (p - w) \cdot \Pr[v > p] = (p - w)(1 - p),$$

a strictly concave function of p that is maximized at

$$p^*(w) = \frac{1 + w}{2}$$

This optimal ask price depends on Anne's own valuation w: the more she likes the object, the higher will her ask-price be.

(c) In the double-auction, we need to verify that the proposed strategy pair constitutes a Nash equilibrium. A pure strategy for Anne is a function $p_A : [0, 1] \to [0, 1]$ that to each possible valuation w that she might have assigns an ask price $p_A(w)$. Likewise, a pure strategy for Bert is a function $p_B : [0, 1] \to [0, 1]$ that to each possible valuation v that he might have assigns a bid price $p_B(v)$. We first note that trade only takes place when $p_A(w) \leq p_B(v)$. Under the proposed strategy profile, this happens iff $2(v - w)/3 > 1/4 - 1/12$ or, equivalently, iff $v - w > 1/4$. From the players' strategies we also have that Anne never asks a price below 1/4, and Bert never bids more than 3/4. In order to verify whether the proposed strategy pair is a Nash equilibrium, we need to consider the optimality of each player's strategy choice, given the other party's strategy. First, given B's strategy: is A's strategy optimal? A takes as given that

$$p_B(v) = \begin{cases} 2v/3 + 1/12 & \text{if } v > 1/4 \\ v & \text{otherwise} \end{cases}$$

Conditional upon her own valuation, w, A sets her price, p, so as to maximize her expected utility. For any given w, her expected utility is

$$u_A(p) = \mathbb{E}\left[\frac{p + p_B(v)}{2} - w \mid p_B(v) > p\right] \cdot \Pr[p_B(v) > p]$$

i. For $0 \leq p \leq 1/4$:

$$\begin{aligned} u_A(p) &= \int_p^{1/4} \left(\frac{p+v}{2} - w\right) dv + \int_{1/4}^1 \left(\frac{p + 2v/3 + 1/12}{2} - w\right) dv \\ &= (p/2 - w)(1-p) + \frac{1}{2}\int_p^{1/4} v dv + \frac{1}{3}\int_{1/4}^1 v dv \end{aligned}$$

ii. For $1/4 < p \leq 1$:

$$\begin{aligned} u_A(p) &= \int_{3p/2-1/8}^1 \left(\frac{p + 2v/3 + 1/12}{2} - w\right) dv \\ &= (p/2 + 1/24 - w)(1 - 3p/2 + 1/8) + \frac{1}{3}\int_{3p/2-1/8}^1 v dv \end{aligned}$$

iii. Taking the derivative with respect to p, we find that, for $0 \leq p \leq 1/4$:

$$u_A'(p) = (1-p)/2 - (p/2 - w) - p/2 = w + 1/2 - p > 0$$

so A never wants to ask a price $p \leq 1/4$. Again taking the derivative with respect to p, but this time for $1/4 < p \leq 1$, we find that:

$$\begin{aligned} u_A'(p) &= (1 - 3p/2 + 1/8)/2 - 3(p/2 + 1/24 - w)/2 - (3p/2 - 1/8)/2 \\ &= 3w/2 - 9p/4 + 9/16 \end{aligned}$$

Hence, $u_A''(p) < 0$ so u_A is a strictly concave function, with unique maximum where $u_A'(p) = 0$, or

$$p = 2w/3 + 1/4$$

proving that A's strategy is the optimal one against B's.

iv. With a similar calculation one can verify that B's strategy is optimal against A's.