COMBINATORIAL GAME THEORY

JONAS SJÖSTRAND

1. What is a combinatorial game?

As opposed to classical game theory, combinatorial game theory deals exclusively with a specific type of two-player games. Informally, these games can be characterized as follows.

(1) There are *two players* who *alternate moves*.
(2) There are *no chance devices* like dice or shuffled cards.
(3) There is *perfect information*, i.e. all possible moves and the complete history of the game are known to both players.
(4) The game will eventually come to an end, even if the players do not alternate moves.
(5) The game ends when the player in turn has no legal move and then he loses.

The last condition is called the *normal play convention* and is sometimes replaced by the *misère play convention* where the player who makes the last move loses. In this course, however, we will stick to the normal play convention.

The players are typically called Left and Right.

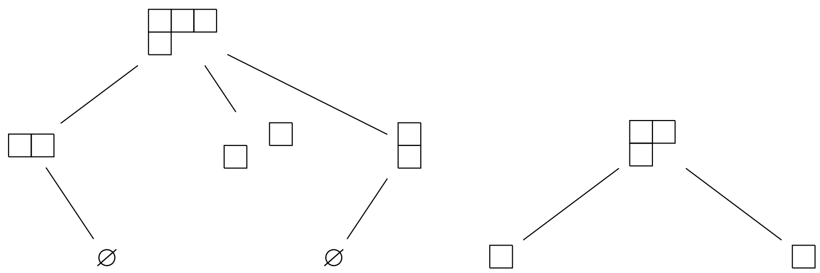
1.1. **Examples of games.**

1.1.1. *Domineering.* A position in *Domineering* is a subset of the squares on a grid. Left places a domino to remove two adjacent vertical squares. Right places horizontally.

1.1.2. *Hackenbush.* A position in Hackenbush is a graph where one special vertex is called the *ground*. In each move, a player cuts an edge and removes any portion of the graph no longer connected to the ground.

In *Blue-Red Hackenbush* (also known as *LR-Hackenbush* or *Hackenbush restrained*) the edges are coloured blue and red, and Left may only cut blue edges, Right only red edges.

1.2. **Game trees.** We may visualize a game as a tree with the start position at the top and where Left follows down-left edges and Right follows down-right edges. Here are the game trees for three Domineering positions.



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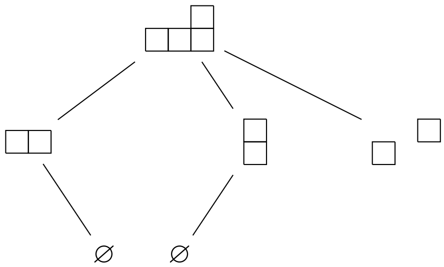


1.3. **Formal definition of games.** From now on, we will refer to game positions simply as *games*.

When developing an abstract theory for games, we have no reason to distinguish between games with isomorphic game trees (like the Domineering games and above) since those are really just different symbolic representations of the same game. In other words, it is the game tree itself that is important and not how we label the vertices or edges. Since a game tree is completely determined by its left and right subtrees, we may simply define a game as a set of left subgames together with a set of right subgames, and indeed we do:

Axioms for games

1. A game is an ordered pair $G = \{\mathcal{G}^L | \mathcal{G}^R\}$ of sets of games. ($\mathcal{G}^L$ and $\mathcal{G}^R$ are the sets of *left* and *right options* of G, respectively.)
2. Two games G and H are *identical*, and we write $G \equiv H$, if and only if $\mathcal{G}^L = \mathcal{H}^L$ and $\mathcal{G}^R = \mathcal{H}^R$.
3. There is no infinite sequence of games $G_1, G_2, \dots$ such that G_{i+1} is a (left or right) option of G_i for all i.

The third axiom guarantees that a game must eventually come to an end, even if the players do not alternate moves. In other words, the game tree has a finite depth. Note however that the sets of options does not need to be finite, so we allow games with infinitely many legal moves.

Note that we use the symbol "$\equiv$" for identical games. The reason is that later on we will define *equivalence* of games and use the ordinary equality sign "$=$" for that.

A typical left or right option of G will be denoted by G^L and G^R, respectively, and we will always omit the curly braces around the sets of options and write $G = \{G_1^L, G_2^L, \dots | G_1^R, G_2^R, \dots\}$.

The simplest of all games is $\{|\}$ which we call *the zero game* and denote by 0. Using 0 we can construct the three games

$$
\begin{aligned}
1 &:= \{0|\}, \\
-1 &:= \{|0\}, \text{ and} \\
* &:= \{0|0\},
\end{aligned}
$$

whose names will be justified later.

In the creation story of games, the zero game was born on day 0 and 1, -1 and $*$ were born at day one. The *birthday* of a game is defined recursively as 1 plus the maximal birthday of its options, with the zero game having birthday 0.

1.4. **Conway induction.**

Theorem 1.1. *Let P be any property that a game can have. If a game is P whenever any option is P, then all games are P.*

More generally: Let P be a property that an n-tuple of games can have. Suppose that, for any n-tuple $G_1, \dots, G_n$, if P holds when any game in the tuple is replaced by one of its options, then the tuple $G_1, \dots, G_n$ is P. Then, any n-tuple of games is P.

Proof. Suppose a game is P whenever any option is. If there is a game G_1 that is not P, then it has an option G_2 that is not P, which has an option G_3 that is not P, and so on. This infinite sequence violates the third axiom of games.

The more general version is proved in a similar way. □

2. The outcome of a game

We always assume optimal play and when we say that a player *wins* a game we have that assumption in mind.

Let us examine the outcomes of the four simplest games 0, 1, -1 and $*$: The game 0 is a win for the second player since the first player loses immediately; Left wins 1 and Right wins -1 whoever starts; and $*$ is a win for the first player.

A moment's thought reveals that any game belongs to exactly one of these four outcome classes.

If G and H belong to the same outcome class we write $G \sim H$, and if, whoever starts, the outcome of G is at least as good for Left as the outcome of H, we write $G \gtrsim H$. Clearly, $1 \gtrsim 0 \gtrsim -1$ and $1 \gtrsim * \gtrsim -1$ while $0 \not\gtrsim * \not\gtrsim 0$ so the Hasse diagram of outcome classes looks like this:

$$\begin{tikzcd}[arrows=dash]
 & 1 \arrow[dl] \arrow[dr] & \\
0 \arrow[dr] & & * \arrow[dl] \\
 & -1 &
\end{tikzcd}$$

Note that

$$G \gtrsim 0 \Leftrightarrow \text{ Left wins as second player, and}$$
$$G \lesssim 0 \Leftrightarrow \text{ Right wins as second player.}$$

3. Sum and negation of games

For convenience, if G is a game and $\mathcal{H}$ is a set of games, let $G + \mathcal{H}$ denote the set $\{G + H \colon H \in \mathcal{H}\}$. Also, let $-\mathcal{H}$ denote the set $\{-H \colon H \in \mathcal{H}\}$.

Definition 3.1 (Sum of games)**.** *For any two games G and H, we define their* sum *$G + H$ as the game where G and H are played in parallel, and the player about to move must choose to make a move in* either *the G-component or the H-component. Formally,*

$$G + H = \{(\mathcal{G}^L + H) \cup (G + \mathcal{H}^L) | (\mathcal{G}^R + H) \cup (G + \mathcal{H}^R)\}.$$

Theorem 3.2. *Addition is commutative and associative, i.e. for any games G, H, and K, we have*

$$G + H \equiv H + G,$$
$$(G + H) + K \equiv G + (H + K).$$

Proof. Intuitively, this is obvious. To find a formal proof is a simple exercise in Conway induction. □

Corollary 3.3. *The class* **Pg** *of partizan games forms an Abelian monoid under addition.*

Definition 3.4 (Negation)**.** *The* negation $-G$ *of a game* G *is the game where the roles of Left and Right are swapped. Formally,*

$$-G = \{-\mathcal{G}^R | -\mathcal{G}^L\}.$$

We write $G - H$ *for the game* $G + (-H)$*.*

Note that $G - G \not\equiv 0$ unless $G \equiv 0$, so the notation seems a bit misleading for the moment. However, things will get better in the next section.

Theorem 3.5. *For any games* G *and* H*, we have*

$$\begin{aligned} -(-G) &\equiv G, \\ -(G+H) &\equiv (-G) + (-H), \\ G \gtrsim H &\Leftrightarrow -G \lesssim -H. \end{aligned}$$

Proof. Again, this is intuitively obvious and a formal proof is a simple exercise in Conway induction. □

4. Comparing games

As combinatorial game theorists, we are primarily interested in outcome classes of games and not so much in the games themselves. However, the outcome class of $G + H$ is not determined by the outcome classes of G and H; for instance, $\{1|\} \sim 1$, but $\{1|\} - 1 \nsim 1 - 1$. We want to consider G and H as "equal" only if they behave similarly when added to other games, and this leads us to the following definition.

Definition 4.1. *Two games* G *and* H *are said to be* equivalent *(or simply* equal*), and we write* $G = H$*, if* $G + K \sim H + K$ *for any game* K*.*

Furthermore, we write $G \geq H$ *if* $G + K \gtrsim H + K$ *for any game* K*.*

Theorem 4.2. *Equality of games is an equivalence relation and* $\geq$ *is a partial order modulo equivalence. In other words, for any games* G*,* H*, and* K*, the following holds.*

$$G = G \text{ (reflexivity)}, \tag{1}$$

$$G = H \Rightarrow H = G \text{ (symmetry)}, \tag{2}$$

$$G = H = K \Rightarrow G = K \text{ (transitivity)}, \tag{3}$$

$$G = H \Rightarrow G \geq H \text{ (reflexivity)}, \tag{4}$$

$$G \geq H \geq G \Rightarrow G = H \text{ (antisymmetry)}, \tag{5}$$

$$G \geq H \geq K \Rightarrow G \geq K \text{ (transitivity)}. \tag{6}$$

Proof. Easy exercise! □

Theorem 4.3. *The partial order and equality is compatible with addition and negation in the sense that, for any games* G*,* H*, and* K*,*

$$\begin{aligned} G = H &\Rightarrow G + K = H + K, \\ G = H &\Rightarrow -G = -H, \\ G \geq H &\Rightarrow G + K \geq H + K, \\ G \geq H &\Rightarrow -G \leq -H. \end{aligned}$$

Proof. Easy exercise! □

Okay, so we have defined equivalence classes of games and a partial order on them which behave properly. What can we say about the algebraic structure of our creation?

Our first task is to characterize the games that compare with zero. To this end we need a lemma.

Lemma 4.4. *$G \gtrsim 0$ and $H \gtrsim 0$ implies $G + H \gtrsim 0$.*

Proof. If Right starts from $G + H$ he must move either to some $G^R + H$ or to some $G + H^R$. By symmetry, without loss of generality we may assume that he moves to $G^R + H$. Since Right has no winning move from $G \gtrsim 0$, there must exist some left option $G^{RL} \gtrsim 0$ of G^R. By induction, $G^{RL} + H \gtrsim 0$ so Left will win the game. □

It turns out that the games that are equal to zero are exactly those won by the second player, the games that are ≥ 0 are those without a winning move for Right, and the games are ≤ 0 are those without a winning move for Left.

Theorem 4.5.

$$\begin{aligned} G \geq 0 &\Leftrightarrow G \gtrsim 0, \\ G \leq 0 &\Leftrightarrow G \lesssim 0, \\ G = 0 &\Leftrightarrow G \sim 0. \end{aligned}$$

Proof. By symmetry, it suffices to show that $G \geq 0 \Leftrightarrow G \gtrsim 0$.

If $G \geq 0$ then $G + K \gtrsim K$ for any game K, and in particular $G + 0 \gtrsim 0$.

For the converse, suppose $G \gtrsim 0$ and let us show that $G + K \gtrsim K$ for any game K.

Informally, if Left has a winning strategy from K, then she can transform it to a winning strategy from $G + K$ by the additional rule that she will only play in the G component as a direct answer to a move there. In that way, she will be acting as the second player in G and can take advantage of her winning strategy there.

Formally, we may reason as follows.

If $K \not\lesssim 0$, i.e. Left has a winning move from K to some K^L, then she also has a winning move from $G + K$ to $G + K^L$ since $G + K^L \gtrsim K^L$ by induction. If $K \gtrsim 0$, i.e. Right has no winning move from K, then, by Lemma 4.4, he has no winning move from $G + K$ either. The implications $K \not\lesssim 0 \Rightarrow G + K \not\lesssim 0$ and $K \gtrsim 0 \Rightarrow G + K \gtrsim 0$ together yield that $G + K \gtrsim K$. □

What if $G - G$ is a win for the second player? Then, by the above theorem, it would be equal to zero and that would tranform our boring Monoid into an Abelian Group!

Theorem 4.6. *$G - G = 0$ for any game G.*

Proof. By the previous theorem, it suffices to show that $G - G \sim 0$, and, by symmetry, it suffices to show that $G - G \gtrsim 0$, that is, Right has no winning move from $G - G$.

Right's move to $G^R - G$ is countered by $G^R - G^R$ which is $\gtrsim 0$ by induction, so Left will win. Analogously, Right's move to $G - G^L$ is countered by $G^L - G^L \gtrsim 0$.

The intuition behind is that the second player can always mimic the previous move in the other component. □

Corollary 4.7. *The class* **Pg** *of partizan games is a partially ordered Abelian Group under addition modulo equality. The identity element is (the equivalence class of)* 0*.*

For convenience, we introduce the notation $G \triangleright H$ as a synonym for $G \not\leq H$, and we write $G \parallel H$ (pronounced "G is fuzzy to H") if $G \triangleright H \triangleright G$, i.e. G and H are not related by the partial order $\geq$.

The following table summarizes the interpretations of comparisons of a game with zero.

$$\begin{aligned} G \geq 0 &\Leftrightarrow \text{ Left wins as second player,} \\ G \leq 0 &\Leftrightarrow \text{ Right wins as second player,} \\ G \triangleright 0 &\Leftrightarrow \text{ Left wins as first player,} \\ G \triangleleft 0 &\Leftrightarrow \text{ Right wins as first player,} \\ G = 0 &\Leftrightarrow \text{ the second player wins,} \\ G \parallel 0 &\Leftrightarrow \text{ the first player wins,} \end{aligned}$$

Theorem 4.8. $G^L \triangleleft G \triangleleft G^R$ *hold for any left option* G^L *and any right option* G^R *of any game* G*.*

Proof. Right has a winning move from $G^L - G$ to $G^L - G^L = 0$ and a winning move from $G - G^R$ to $G^R - G^R = 0$. □

Theorem 4.9. $G \geq H$ *if and only if all* $G^R \triangleright H$ *and all* $H^L \triangleleft G$*.*

Proof. $G \geq H \Leftrightarrow G - H \geq 0 \Leftrightarrow G - H \gtrsim 0$ which means that Right has no winning move from $G - H$. But Right's move to $G^R - H$ is winning if and only if $G^R - H \leq 0$ and Right's move to $G - H^L$ is winning if and only if $G - H^L \leq 0$. □

Theorem 4.10. *Let* $G = \{\mathcal{G}^L | \mathcal{G}^R\}$ *and* $H = \{\mathcal{H}^L | \mathcal{H}^R\}$ *be games and suppose there are bijections* $\phi_L \colon \mathcal{G}^L \to \mathcal{H}^L$ *and* $\phi_R \colon \mathcal{G}^R \to \mathcal{H}^R$ *with the property that* $\phi_L(G^L) = G^L$ *and* $\phi_R(G^R) = G^R$ *for any options* G^L *and* G^R*. Then,* $G = H$*.*

Proof. For each left option G^L we have $G^L = \phi_L(G^L) \triangleleft H$ and for each right option G^R we have $H \triangleleft \phi_R(G^R) = G^R$, so we conclude that $G \geq H$.

By symmetry, $H \geq G$. □

Theorem 4.11. $G \triangleright H \geq K \Rightarrow G \triangleright K$ *and* $G \geq H \triangleright K \Rightarrow G \triangleright K$*.*

Proof. Simple exercise! □

5. Simplifying games

Theorem 5.1. *The value of* G *is unaltered or increased when we*

(1) *increase any* G^L *or* G^R*, or*
(2) *remove some* G^R *or add a new* G^L*.*

Proof. Let G' be the game obtained by so modifying G. Then in the game $G - G'$ it is easy to check that Right has no good first move. □

Definition 5.2. *Let* G *a game. A left option* G^L *is* dominated *by another left option* $G^{L'}$ *if* $G^{L'} \geq G^L$*. Similarly, a right option* G^R *is dominated by another right option* $G^{R'}$ *if* $G^{R'} \leq G^R$*.*

Theorem 5.3. *Removing a dominated option does not alter the value of the game.*

Proof. Let G be a game with a left option G^L dominated by another left option $G^{L'}$, and let G' be the game obtained by removing the option G^L from G. That $G \geq G'$ follows from Theorem 5.1. We will prove that $G \leq G'$ by showing that Left has no good first move in $G - G'$. Each left option in $G - G'$ has a right counterpart in the other component, except $G^L - G'$. But this is countered by $G^L - G^{L'}$ which is ≤ 0, so Right will win.

The corresponding statement for dominated right options follows by symmetry. □

Definition 5.4. *Let G be a game. A left option G^L of G is said to be* reversible (through G^{LR}) *if it has a right option $G^{LR} \leq G$. Similarly, a right option G^R is* reversible (through G^{LR}) *if it has a left option $G^{RL} \geq G$.*

Theorem 5.5. *If G has a left option G^L that is reversible through G^{LR}, then the value of G does not change if we replace G^L by all the left options of G^{LR}.*

Proof. Let G' be the game obtained from G by replacing G^L by all the left options of G^{LR}. We will show that neither player has any good move in $G - G'$.

If Left starts and moves to $G^L - G'$, Right moves to $G^{LR} - G'$. Left can either move to some $G^{LRL} - G'$, in which case Right moves to $G^{LRL} - G^{LRL}$ and wins, or she can move to some $G^{LR} - G^R$, in which case Right wins since $G^{LR} \leq G \lhd G^R$.

If Right starts and moves to some $G - G^{LRL}$, Left will win since $G \geq G^{LR} \rhd G^{LRL}$. Any other first move for Right has counterparts for Left in the other component. □

Definition 5.6. *A game is in* canonical form *if it has no dominated or reversible options and if all its options are in canonical form.*

Theorem 5.7. *Every equivalence class of short games has exactly one representative in canonical form. In other words, for each short game G there is a unique game G' in canonical form such that $G = G'$.*

Proof. Since both ways of simplifying games (removing dominated options and bypassing reversible options) reduce the number of positions, we must eventually reach a game that cannot be simplified further. This proves existence. To prove uniqueness, we assume that G and H are two equal games in canonical form. We have to show that $G \equiv H$. Let G^L be some left option of G. Since $G^L \lhd G = H$, there must be a right option $G^{LR} \leq H$ or a left option H^L such that $G^L \leq H^L$. The first is impossible since G^L is not reversible. Similarly, there is some $G^{L'}$ such that $H^L \leq G^{L'}$, so $G^L \leq G^{L'}$. But there are no dominated options either, so $G^L = H^L = G^{L'}$. By induction, $G^L \equiv H^L$. In that way, we see that G and H have the same set of (identical) left options, and the same is true for the right options. □

6. Numbers

By $x < y$ we mean that $x \leq y$ and $x \neq y$, just as expected.

Definition 6.1. *A game x is said to be a* number *if*

- *all options of x are numbers, and*
- *$x^L < x^R$ for any left option x^L and any right option x^R of x.*

For instance, 0, 1 and -1 are numbers, but $*$ is not. We usually denote numbers by the lower-case letters x, y, z etc.

Note that a game might *equal* a number without actually *being* one. Find such a game as an exercise!

Theorem 6.2. *The class of numbers is closed under addition and negation.*

Proof. Conway induction! $\square$

Theorem 6.3. *If x is a number then $x^L < x < x^R$ for any x^L, x^R.*

Proof. Since $x^L \lhd x \lhd x^R$, it suffices to show that $x^L \le x \le x^R$.

Right's move from $x - x^L$ to $x^R - x^L$ is bad since $x^R > x^L$ by the definition of a number, and Right's move from $x - x^L$ to $x - x^{LL}$ is bad since $x \rhd x^L > x^{LL}$ by induction. $\square$

Theorem 6.4. *Numbers are totally ordered, that is, for any numbers x and y, either $x \le y$ or $x \ge y$.*

Proof. The inequality $x \not\ge y$ implies either some $x^R \le y$ or $x \le$ some y^L, whence either $x < x^R \le y$ or $x \le y^L < y$. $\square$

The following theorem states that if there is a number fuzzily in between the left and right options of a game, then the game equals the *simplest* such number, that is, the unique such number that has no option with the same property.

Theorem 6.5 (The simplicity theorem)**.** *Suppose G is a game and x is a number such that*

- *$G^L \lhd x \lhd G^R$ for any left option G^L and right option G^R of G, and*
- *no option of x satisfies the same condition.*

Then $G = x$.

Proof. We have $G \ge x$ unless some $G^R \le x$ (which is false) or $G \le$ some x^L. But from $G \le x^L$ we can deduce $G^L \lhd G \le x^L < x \lhd G^R$ for all G^L, G^R, from which we have $G^L \lhd x^L \lhd G^R$, contradicting the supposition about x. So $G \ge x$, similarly $G \le x$, and so $G = x$. $\square$

6.1. **Dyadic rationals.** We have already defined the numbers 0, 1 and -1, and we may define any integer by adding ones or minus ones, for instance $5 := 1+1+1+1+1$.

Theorem 6.6. *For any integer n, we have $n = \{\, n-1 \mid \,\}$ if n is positive and $n = \{\, \mid n+1 \,\}$ if n is negative.*

We leave the proof as an exercise.

Multiplication of a number (or, indeed, any game) by an integer can also be defined just by iterated addition, for instance $3x := x + x + x$. But is it possible to divide a number by an integer?

Let us first observe that if a quotient exists it must be unique (up to equality). Indeed, if m is a non-zero integer and $mx = my = z$ then $m(x - y) = 0$ and it follows that $x - y = 0$. (Can you see why?) Hence, z/m is well-defined if it exists. Also, for quotients that exist, ordinary algabraic rules apply:

- If z/m and w/m exist, then the quotient $(z + w)/m = z/m + w/m$ exists.
- If z/m exists, then, for any non-zero integer k, $(kz)/m$ exists and equals $k(z/m)$, and kz/km exists and equals z/m.

As an exercise, show that $1/2$ exists and equals $\{\, 0 \mid 1 \,\}$.

Theorem 6.7. *For any integer $n \ge 1$, the fraction $1/2^n$ exists and equals $\{\, 0 \mid 1/2^{n-1} \,\}$.*

Proof. Let $x = 1/2^{n-1}$ and $y = \{\,0\,|\,x\,\}$. By induction, $x = \{\,0\,|\,2x\}$. Clearly,

$$0 < y < x < x + y < 2x,$$

so x, but no option of x, lies between y and $x+y$. Thus, by the simplicity theorem, $x = \{\,y\,|\,x+y\,\}$ which equals $2y$ by the definition of addition. $\square$

It follows that any quotient of an integer by a power of two exists. Such fractions are called *dyadic rationals.*

Theorem 6.8. *For any nonnegative integer $n \geq 0$ and any odd integer k, we have*

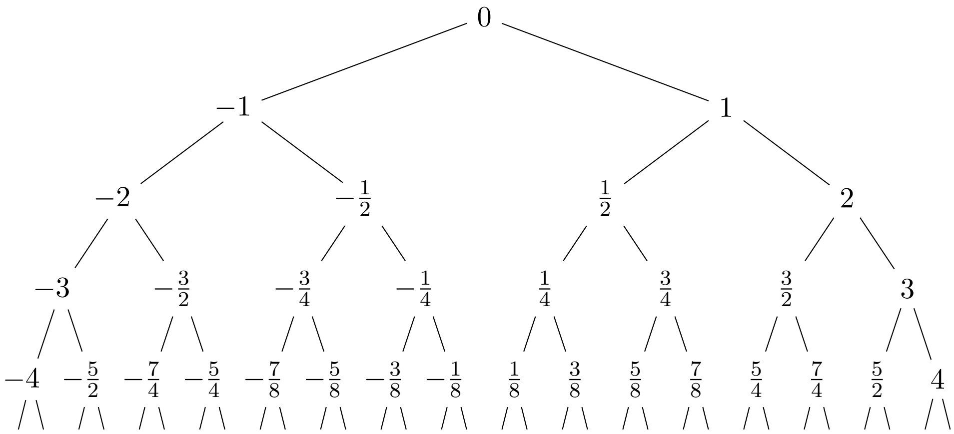
$$\frac{k}{2^n} = \left\{ \frac{k-1}{2^n} \,\middle|\, \frac{k+1}{2^n} \right\}. \tag{7}$$

Proof.

$$\begin{aligned}
\frac{k}{2^n} &= \frac{1}{2^n} + \cdots + \frac{1}{2^n} \\
&= \left\{ \frac{k-1}{2^n} + 0 \,\middle|\, \frac{k-1}{2^n} + \frac{1}{2^{n-1}} \right\} \\
&= \left\{ \frac{k-1}{2^n} \,\middle|\, \frac{k+1}{2^n} \right\}.
\end{aligned}$$

$\square$

The following picture shows how all numbers form an infinite tree. Each node has two children, namely the first later numbers born just to the left and right of it.



Every number x can be found in the infinite tree, and the *sign-expansion* of x tells us how to reach x if we start at the top node 0 and walk along the edges down the tree. The sign-expansion is a (possibly) infinite sequence of pluses and minuses, where plus means right and minus means left. For instance, the sign-expansion of $7/4$ is $++-+$.

Numbers are the coolest class of games in the sense that no player wants to be the first to make a move in a number. The following theorem shows that, in order to win a game with several components, you do not have to move in a number unless there is nothing else to do.

Theorem 6.9 (Weak number avoidance theorem)**.** *If G is a game that is not equal to a number and x is a number, then*

$$G + x \rhd 0 \iff \textit{some } G^L + x \geq 0.$$

Proof. Suppose on the contrary that Left has a winning move to some $G + x^L$ but not to any $G^L + x$. Then

$$\text{all } G^L \lhd -x < -x^L \leq G \lhd \text{all } G^R,$$

where the second inequality follows from the definition of a number and the last inequality follows from Theorem 4.8. But now the simplicity theorem yields that G is a number, which contradicts the assumption in the theorem. □

7. Blue-Red Hackenbush is always a number

Lemma 7.1. *In Blue-Red Hackenbush, on chopping a blue edge, the value stricly decreases; on chopping a red one it strictly increases.*

Proof. Let G^L be the result after chopping a blue edge e from a position G. If Right starts in the game $G - G^L$, Left can win by mimicking Right's move in the other component until Right chops an edge in the G-component that has no counterpart in the $-G^L$-component. If this happens, Left simply chops the edge e from the G-component, thereby obtaining a zero game. This shows that $G - G^L \geq 0$ and thus $G^L < G$ (since an option never is equal to the game itself).

The argument for $G < G^R$ is completely analogous. □

Now the following theorem follows by induction.

Theorem 7.2. *In Blue-Red Hackenbush every position is a number.*

7.1. **Trees.** There is a simple rule for computing the Blue-Red Hackenbush value of a tree.

For a Hackenbush tree x with the root on the ground, let $1 : x$ (and $-1 : x$) denote the tree obtained by inserting a blue (red) edge between the ground and the root. Every tree can be written as either $1 : x$ or $-1 : x$, where x is a sum of smaller trees, so in order to compute the value of any tree we only need a method to compute the value of $1 : x$ given the value of x.

The moves from $1 : x$ are to 0 and $1 : x^L$ for Left and to $1 : x^R$ for Right, and the moves from $-1 : x$ are to $-1 : x^L$ for Left and to 0 and $-1 : x^R$ for Right. Now, the function $1 : x = \{ 0, 1 : x^L \mid 1 : x^R \}$ maps all numbers onto positive numbers in order of simplicity. Thus 0, the simplest number, maps to 1, the simplest positive number. Then -1 and 1 map to the simplest positive numbers to the left and right of 1, namely 1/2 and 2 respectively, and so on. In terms of sign-expansions, $1 : x$ is obtained by inserting a + at the beginning of the sign-expansion of x, and $-1 : x$ is obtained by inserting a minus sign at the beginning.

8. Comparing games with numbers

A game is *short* if it has only finitely many positions altogether. Short games are bounded in the following sense.

Theorem 8.1. *For any short game G there is some integer n such that $-n < G < n$.*

Proof. Take n greater than the total number of positions of G and consider playing in $G+n$. Left can win this by just decreasing n by 1 each time he moves, waiting for Right to run himself down in G. Since $G+n>0$, we have $G>-n$, and similarly $G<n$. □

Definition 8.2. *A* stop *is a symbol of the form x_+ or x_-, where x is a number. The union of stops and numbers is totally ordered, larger numbers being larger and the subscript sign being used for tie-breaks, so for any numbers $x<y$ we have*

$$x_- < x < x_+ < y_- < y < y_+.$$

Definition 8.3. *Define the left stop $L(G)$ and the right stop $R(G)$ of a short game G by the rule*

$$L(G) := \begin{cases} x_- & \text{if } G \text{ is equal to the number } x, \\ \max_{G^L} R(G^L) & \text{if } G \text{ is not equal to a number,} \end{cases}$$

$$R(G) := \begin{cases} x_+ & \text{if } G \text{ is equal to the number } x, \\ \min_{G^R} L(G^R) & \text{if } G \text{ is not equal to a number.} \end{cases}$$

To see that the maximum and the mininum in the definition are well-defined, note that if a short game has no left option or no right option, by the simplicity theorem and Theorem 8.1, it must be a number.

Theorem 8.4. *Let G be a short game. For any number x the following equivalences hold.*

$$L(G) > x \iff G \rhd x$$
$$R(G) < x \iff G \lhd x.$$

Proof. If G is a number, then $L(G)=G_-$ and $R(G)=G_+$, and the statements are true. If G is not a number, we have the following equivalences.

$$\begin{aligned} L(G) > x \ &\Leftrightarrow\ \text{some } R(G^L) > x && \text{(by definition of left stop)} \\ &\Leftrightarrow\ \text{some } G^L \geq x && \text{(by induction)} \\ &\Leftrightarrow\ G \rhd x && \text{(by the weak number avoidance theorem).} \end{aligned}$$

The equivalence $R(G) < x \ \Leftrightarrow\ G \lhd x$ is proved analogously. □

Corollary 8.5. *Let G be a short game. Then, for any number x, the following equivalences hold.*

$$\begin{aligned} x \geq G &\iff x > L(G), \\ x \parallel G &\iff L(G) > x > R(G), \\ G \geq x &\iff R(G) > x. \end{aligned}$$

Furthermore, if $G=H$ then $L(G)=L(H)$ and $R(G)=R(H)$.

Theorem 8.6. *A short game G is equal to a number if and only if $R(G^L) < L(G^R)$ for all G^L and G^R. Furthermore, if G is equal to a number, then it equals the simplest number between all $R(G^L)$ and all $L(G^R)$.*

Proof. First, suppose G is equal to a number x. Then, for any G^R we have $G^R \rhd x$ which implies that $L(G^R) > x$ according to Theorem 8.4. In the same way we can prove that $R(G^L) < x$ for any G^L, and it follows that $R(G^L) < L(G^R)$.

Now, suppose $R(G^L) < L(G^R)$ for all G^L and G^R and let x be the simplest number in between. We will prove that $G - x = 0$ by showing that all moves from $G - x$ are bad. Left's move to $G^L - x$ is bad since $R(G^L) < x$ implies that $G^L \lhd x$ by Theorem 8.4. Consider Left's move to $G - x^R$. Since x is the simplest number between all $R(G^L)$ and all $L(G^R)$ there exists a G^R such that $L(G^R) < x^R$, and Right counters by moving to $G^R - x^R$ for such a G^R. By Theorem 8.4, $G^R \le x^R$ so Right will win the game. Since Left has no good move from $G - x$, we conclude that $G - x \le 0$. Similarly, we can show that $G - x \ge 0$ and thus $G = x$. □

Lemma 8.7. *If G is a short game that is not equal to a number, then there is a left option G^L such that $G^L - G$ is greater than any negative number.*

Proof. Choose G^L such that $R(G^L) = L(G)$ and let ε be any positive number. Write ε as a sum of two positive numbers $\varepsilon = \varepsilon_1 + \varepsilon_2$. This can always be done, for instance by letting $\varepsilon_1 = \{\,0 \,|\, \varepsilon\,\}$. Now, by Corollary 8.5 we have

$$G^L + \varepsilon_1 > R(G^L) = L(G) > G - \varepsilon_2$$

whence $G^L - G > -\varepsilon$. □

Theorem 8.8 (Translation theorem)**.** *If x is a number and G is a short game that is not equal to a number, then*

$$G + x = \{\,\mathcal{G}^L + x \,|\, \mathcal{G}^R + x\,\}.$$

Proof. Consider a left option of the form $G + x^L$. Since $x^L - x$ is negative, by Lemma 8.7, there is a G^L such that $G^L - G > x^L - x$. Therefore, the option $G + x^L$ is dominated by the option $G^L + x$ and can be removed. An analogous argument applies to $G + x^R$. □

We may add a number to a stop by the rule $y_+ + x := (y + x)_+$ and $(y_- + x) := (y + x)_-$.

Corollary 8.9. *For any short game G and dyadic rational x, we have $L(G + x) = L(G) + x$ and $R(G + x) = R(G) + x$.*

Proof. The case when G is equal to a number is trivial, so suppose G is not equal to a number. Then, by the translation theorem, $L(G + x) = \max_{G^L} R(G^L + x)$ which, by induction, equals $\max_{G^L} R(G^L) + x = L(G) + x$. That $R(G + x) = R(G) + x$ is shown analogously.

Note that we used the fact that the left and right stops respect equality of games. □

9. Thermography — cooling games down

A game is called *hot* if the left stop is strictly greater than the right stop. In a hot game, the players are keen to be the first to make a move. Numbers are *cold* — since $x^L < x < x^R$ for numbers, no player will ever want to make any move. (There are also games where the left and right stop coincide; they are sometimes called *tepid* games.) A hot game can be cooled down by introducing a tax on moves.

Definition 9.1. *If G is a short game and t a positive dyadic rational, then we define the* cooled game G_t *by the formula*

$$G_t := \{\, {G^L}_t - t \,|\, {G^R}_t + t \,\}.$$

unless this formula defines a number (which it will for all sufficiently large t). For the smallest values of t for which this happens, the number turns out to be independent of t, and we define G_t to be this number.

We will justify the assertions in the definition later on.

The *thermograph* of a short game G is a diagram plotting the left and right stops of G_t as a function of t, with t increasing vertically, and the stop values increasing to the left. (The subscript signs of the stops can be safely ignored. We will recover them later.)

The following theorem tells us how to construct the thermograph of a game if we have already constructed the thermographs of its options.

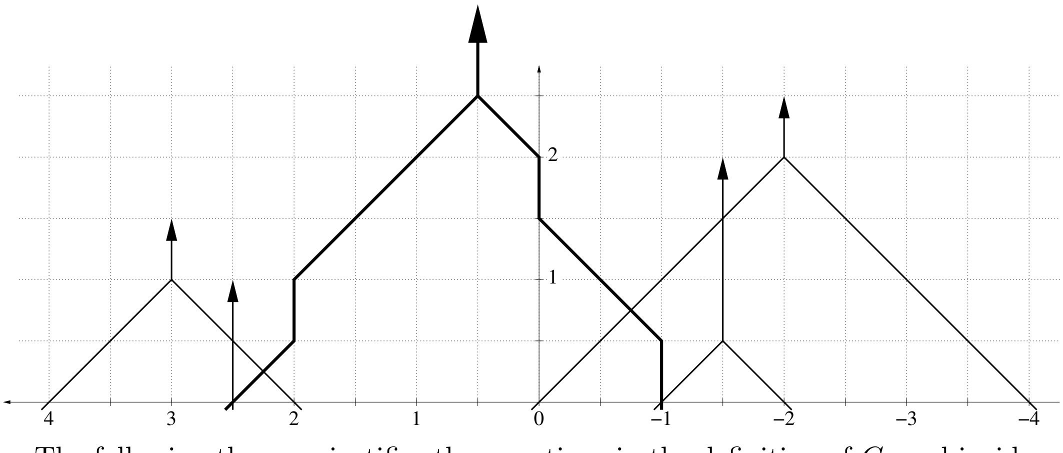
Theorem 9.2. *For all short games G and dyadic rationals $t \geq 0$, we have*

$$L(G_t) = \max_{G^L} R({G^L}_t) - t =: L_t, \text{ say, and}$$
$$R(G_t) = \min_{G^R} L({G^R}_t) + t =: R_t, \text{ say,}$$

unless possibly $L_t < R_t$. In this latter case, G_t is a number x, namely the simplest number between L_u and R_u for all small enough u with $L_u < R_u$, and we then have $L(G_t) = x_-$ and $R(G_t) = x_+$.

Proof. This follows from Corollary 8.9 and Theorem 8.6. For the moment, we are continuing to suppose that G_t is well-defined. □

As an example, here are the thermographs of $G = \{\, \frac{5}{2}, \{4\,|\,2\} \mid \{-1\,|\,-2\}, \{0\,|\,-4\}\,\}$ (thick lines) and of its options (thin lines):



The following theorem justifies the assertions in the definition of G_t and incidentally makes Theorem 9.2 an honest theorem.

Theorem 9.3. *For any short game G, the left border of the thermograph is a line proceeding either vertically or diagonally up and right in stretches, the right boundary being in stretches vertical or diagonal up and left. Beyond some point, both boundaries coincide in a single vertical line — the mast. The coordinates of all corners in the diagram are dyadic rationals.*

Proof. This requires only the observation that on subtracting t from a line which is vertical or diagonal up-and-left we obtain one correspondingly diagonal up-and-right or vertical, and that two such lines aiming towards each other must meet at a point whose coordinates can be found with a single division by 2. □

Theorem 9.4. *The left and right stops $L(G_t)$ and $R(G_t)$ are "just inside" the boundary of the thermograph on vertical stretches, "just outside" on diagonal stretches. At the points of the mast above its foot, $L(G_t) < R(G_t)$. At corners of the diagram the subscript sign is the same as for immediately smaller values of t, so the behaviour is "continuous downwards".*

Proof. These properties are preserved in the passage from the thermographs for G^L and G^R to that for G. □

The height of the root of the mast is called the *temperature* of G and is denoted by $t(G)$. It is equal to $\inf\{t : G_t \text{ equals a number}\}$.

The horizontal coordinate of the mast is called the *mean value* of G and is denoted by G_∞ or $m(G)$. It is the number G eventually becomes when it is frozen.

For our example game above, $t(G) = 5/2$ and $G_\infty = 1/2$.

Lemma 9.5. *For any short game G, any number x and any dyadic rational $t \geq 0$, we have $(G + x)_t = G_t + x$.*

Proof. This follows from the translation theorem. □

Theorem 9.6. *For any games G and H and any dyadic rational t, we have $(G + H)_t = G_t + H_t$.*

Proof. Let s be the smallest of the three temperatures $t(G)$, $t(H)$ and $t(G + H)$. There is a u slightly larger than s such that, for any $r \leq u$,

$$
\begin{aligned}
G_r &= \{\, {G^L}_r - r \mid {G^R}_r + r \,\}, \\
H_r &= \{\, {H^L}_r - r \mid {H^R}_r + r \,\}, \text{ and} \\
(G+H)_r &= \{\, {(G+H)^L}_r - r \mid {(G+H)^R}_r + r \,\}.
\end{aligned}
$$

If $t \leq u$ we have

$$
\begin{aligned}
(G+H)_t &= \{\, (G^L + H)_t - t, (G + H^L)_t - t \mid (G^R + H)_t + t, (G + H^R)_t + t \,\} \\
&= \{\, ({G^L}_t + H_t - t, G_t + {H^L}_t - t \mid {G^R}_t + H_t + t, G_t + {H^R}_t + t \,\} \\
&= G_t + H_t,
\end{aligned}
$$

where the second equality follows from induction. If $t > u$ we have $G_t = (G_u)_{t-u}$, $H_t = (H_u)_{t-u}$ and $(G + H)_t = ((G + H)_u)_{t-u} = (G_u + H_u)_{t-u}$. Since at least one of G_u, H_u and $(G + H)_u$ is a number, it follows from Lemma 9.5 that

$$(G_u + H_u)_{t-u} = (G_u)_{t-u} + (H_u)_{t-u}.$$

□

Theorem 9.7. *For any short games G and H,*

$$
\begin{aligned}
(G+H)_\infty &= G_\infty + H_\infty \\
t(G+H) &\leq \max\{t(G), t(H)\}
\end{aligned}
$$

Proof. The equality follows directly from the preceding theorem.

Suppose $t = t(G + H) > \max\{t(G), t(H)\}$. Then G_t and H_t would be numbers but not $(G + H)_t = G_t + H_t$ — a contradiction. □

Theorem 9.8. *For any short game G, the inequalities*

$$G_\infty - t(G) - \varepsilon < G < G_\infty + t(G) + \varepsilon$$

hold for any dyadic rational $\varepsilon > 0$.

Proof. This follows from Corollary 8.5 and the fact that the borders of the thermograph of G has at least a 45 degree slope. □

Theorem 9.9 (The mean value theorem)**.** *For any short game G and any integer n, the inequalities*

$$nG_\infty - t(G) - \varepsilon < nG < nG_\infty + t(G) + \varepsilon$$

hold for any dyadic rational $\varepsilon > 0$.

Proof. This follows from the above theorem since $(nG)_\infty = nG_\infty$ and $t(nG) \leq t(G)$ by Theorem 9.7. □

10. Impartial games and the game of Nim

A game is called *impartial* if, from any of its positions, both players would have the same legal moves if they were about to play. Formally, a game is impartial if

- all its options are impartial, and
- its set of left options and its set of right options are equal.

An example of a game that is not impartial is chess, since white can only move white chessmen and black can only move black chessmen.

A classical example of an impartial game is the game of Nim, which is played as follows. On a table are a number of piles of sticks. In each move a player chooses one of the piles and removes one or more sticks from it. The player that removes the last stick wins.

If there is only one pile, clearly the first player wins by removing all sticks. If there are two piles things get slightly more complicated: If the piles contain the same number of sticks, the second player wins by mimicking the first player's strategy — when the first player removes some sticks from one of the piles, the second player immediately removes the same number of sticks from the other pile. If the piles contain different numbers of sticks, the first player wins after equalising the piles.

What if there are three or more piles? In 1901, Charles Bouton found the general strategy:

- For two nonnegative integers a and b, define the *nim sum* $a \oplus b$ as the bitwise XOR of a and b when they are written as binary numbers. For instance, if $a = 14 = (1110)_2$ and $b = 5 = (101)_2$ then $c = (1011)_2 = 11$.
- If the nim sum of all piles is zero, the second player wins. If it is positive, the first player wins by a move that makes it zero.

To see that this strategy works, we must check that

- if the nim sum is zero, any move makes it positive, and
- if the nim sum is positive, some move makes it zero.

Suppose the nim sum is zero and consider a move that removes r sticks from a pile with a sticks. Then the new nim sum will not be zero since the binary expansion of $a - r$ is not the same as the binary expansion of a.

For the converse, suppose the nim sum is positive, say q, and let $q_j q_{j-1} \cdots q_0$ be the binary expansion of q. Then, some pile, of size a say, must have a one at position

j in its binary expansion. The move that reduces a to $q \oplus a$ will make the new nim sum zero.

10.1. **A more efficient notation.** As impartial games have the same set of left and right options, our usual notation $G = \{ G^L \,|\, G^R \}$ is redundant, and we will usually identify G by its set of (left or right) options. For instance, instead of $G = \{ *, \{ * \,|\, * \} \,|\, *, \{ * \,|\, * \} \}$ we will simply write $G = \{*, \{*\}\}$.

In an impartial game, either the first or the second player to move will win the game, independent of who is Left and who is Right. If the first player wins it is an $\mathcal{N}$-game (as in the *next* player) and if the second player wins it is a $\mathcal{P}$-game (as in the *previous* player). Recall that $\mathcal{N}$-games are fuzzy to zero and $\mathcal{P}$-games are equal to zero.

For integers $n \geq 0$, let $*n$ denote the game of Nim with a single pile of n sticks, also called a *nimber*. As an example, let us see how the nimber $*3$ is represented as a set.

The options of $*3$ are $*2$, $*1$, and $*0$. The options of $*2$ are $*1$ and $*0$. The game $*1$ has only one option, $*0$, and the game $*0$ has no options at all so it is the empty set $*0 = \emptyset = \{\}$. We get

$$\begin{aligned} *3 &= \{ \quad *2, \quad *1, \quad *0 \quad \} \\ &= \{ \quad \{*1, *0\}, \quad \{*0\}, \quad \{\} \quad \} \\ &= \{ \quad \{\{*0\}, \{\}\}, \quad \{\{\}\}, \quad \{\} \quad \} \\ &= \{ \quad \{\{\{\}\}, \{\}\}, \quad \{\{\}\}, \quad \{\} \quad \}. \end{aligned}$$

Note that $*0 \equiv 0$ and $*1 \equiv *$.

Theorem 10.1. *For any nonnegative integer n, the canonical form of $*n$ is*

$$*n = \{*0, *1, \ldots, *(n-1)\}.$$

Proof. We just have to check that there are no dominated or reversible options, and we leave that as an exercise. □

Note that for any impartial game G, we have $-G \equiv G$ and thus $G + G = 0$; the choice between minus and plus does not matter for impartial games.

10.2. **Grundy values and Grundy's theorem.** We define the *mex* (or *minimum excluded value*) of a finite set of nimbers to be the smallest nimber not in that set.

Theorem 10.2 (Grundy's theorem)**.** *For any finite set G of nimbers, $G = \operatorname{mex} G$ as games.*

Proof. Let $*n$ be the smallest nimber not in G, that is, $\operatorname{mex} G = *n = \{*0, *1, \ldots, *(n-1)\}$. We must show that there are no good moves from $G - *n$.

A move to $G - *k$ for $k < n$ can be countered by $*k - *k$ since $*k \in G$. A move to $*k - *n$ is countered by $*k - *k$ if $k < n$ and by $*n - *n$ if $k > n$. □

So Grundy's theorem shows that every short impartial game is equal to a nimber! (In fact this is true for impartial games in general if we allow infinite ordinal nimbers, but that is not important for us.)

The integer n such that $G = *n$ is called the *Grundy value* of G and is often denoted by $g(G)$. It follows that an impartial game is a $\mathcal{P}$-game if and only if its Grundy value is zero.

The optimal strategy of Nim shows us how to add nimbers:

Theorem 10.3. *For any nonnegative integers m and n, we have*

$$*m + *n = *(m \oplus n).$$

Proof. Since $m \oplus n \oplus (m \oplus n) = 0$, a game of Nim with three piles of sizes m, n, and $m \oplus n$ is a zero game. In other words, $*m + *n + *(m \oplus n) = 0$. $\square$

E-mail address: jonass@kth.se