SF2972 GAME THEORY

Lecture 3: Finite games I

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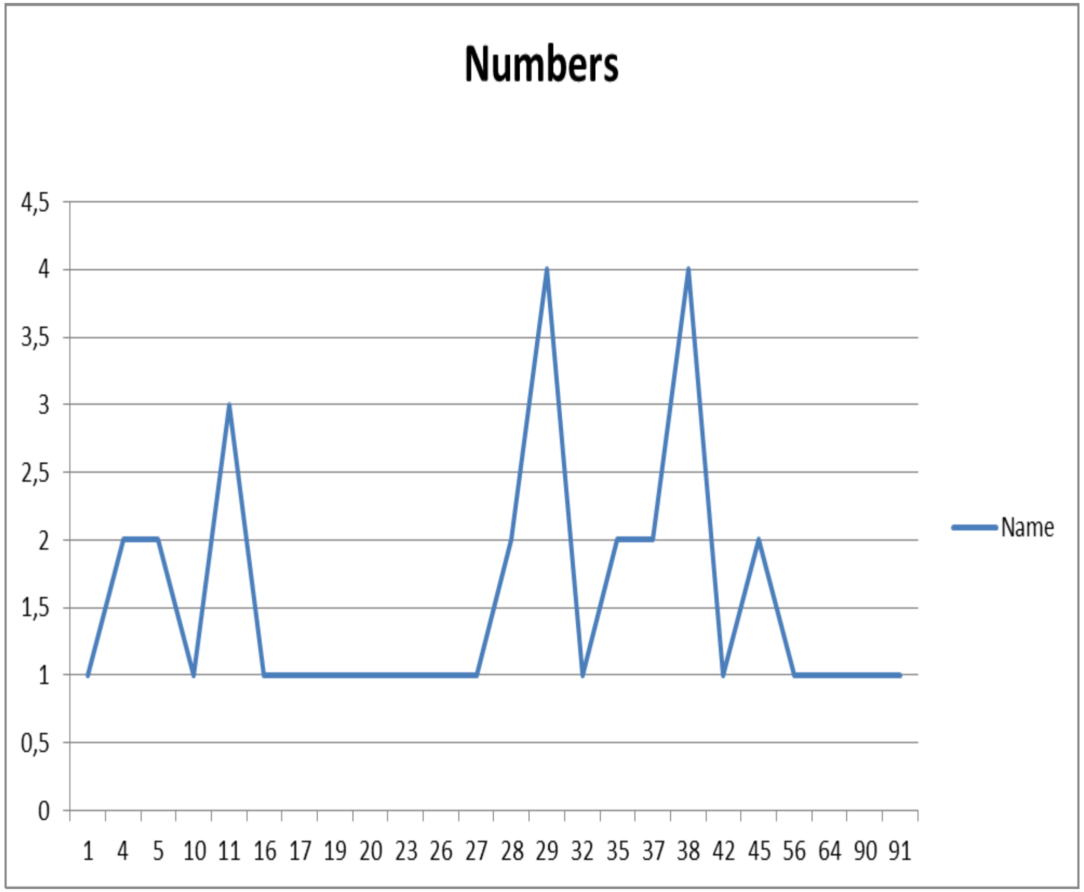
The result of the experiment last lecture:



- We note one peak around 37.5, which is three quarters of 50. This is the best reply if one believes that others' bids are uniformly distributed over the whole strategy set.

- We also note a peak around 28. That is the best reply if one believes the others' bid at 37.5.

- The data shows that rationality and the game are not common knowledge in the class. Instead, "level-k" reasoning, for k=1 and k=2, does a good job in picking the spikes.

- The winning bid was 23.

1 Finite normal-form games

- Recall:

Definition 1.1 *A* **normal-form game** *is a triplet* $G = \langle I, S, u \rangle$ *where*

(a) I *is the set of* **players**

(b) $S = \times_{i \in I} S_i$ *is the set of* **strategy profiles**, *and* S_i *is the* **strategy set** *of player* i

(c) $u : S \to \mathbb{R}^{|I|}$ *is the* **combined payoff function**, *where* $u_i(s) \in \mathbb{R}$ *the* **payoff** *to player* i *when profile* s *is played*

- Such a game is called *finite* if S is finite

- Let $G = \langle I, S, u \rangle$ be any finite game

- For each player $i \in I$ write $S_i = \{1, 2, ..., m_i\}$ for the player's (finite) strategy set

- Suppose that each player can randomize over his or her strategy set if he/she likes

- Then the analysis really concerns what we will call *the mixed-strategy extension* of the given game G, a game $\tilde{G} = \langle I, \boxdot(S), \tilde{u} \rangle$ with the same player set I.

- We proceed to first carefully specify $\boxdot(S)$ and $\tilde{u}$, and then to a general analysis of such games $\tilde{G}$

1.1 Mixed strategies

- The *mixed-strategy set* for player i is the set $\Delta_i = \Delta(S_i)$ of probability distributions over S_i:

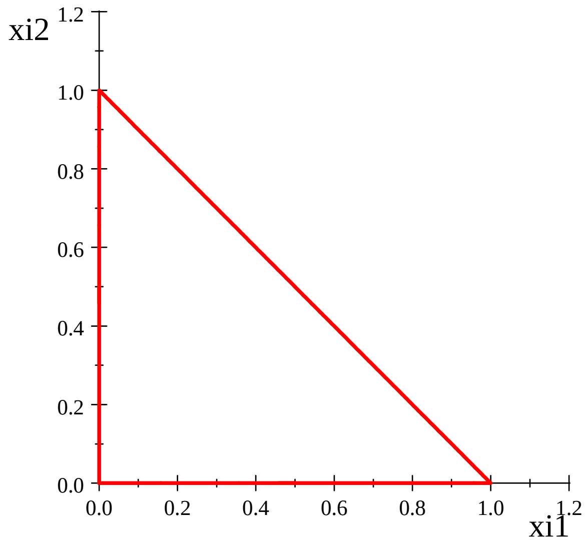
$$\Delta(S_i) = \{x_i \in \mathbb{R}_+^{m_i} : \sum_{h=1}^{m_i} x_{ih} = 1\}$$

where $h = 1, 2, .., m_i \in S_i$ are i's pure strategies. (Hence, for $s_i = h$ we write $x_i(s_i) = x_{ih}$.)

- The *vertices* of Δ_i are the *unit vectors,* $e_i^h \in \mathbb{R}^{m_i}$ with all components except h being zero. We interpret the mixed strategy $e_i^h \in \Delta_i$ as *playing pure strategy* h (using it with probability one)

- The (relative) *interior*: $int(\Delta_i) = \{x_i \in \Delta_i : x_{ih} > 0 \ \forall h \in S_i\}$. These are player i's *interior* or *completely mixed* strategies

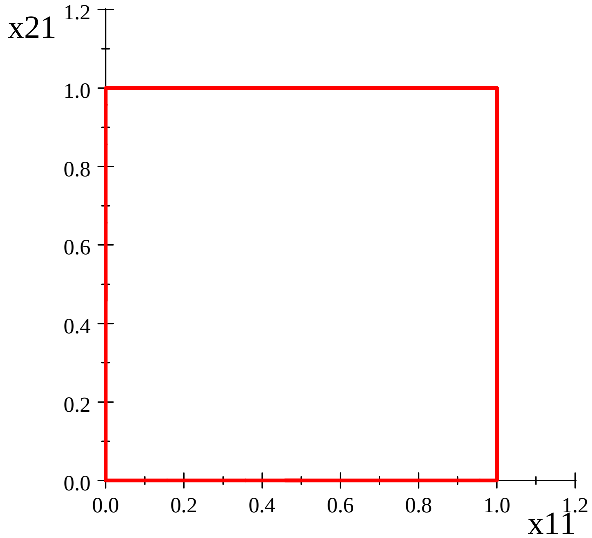
Example: $m_i = 3$



- A mixed-strategy *profile* $x = (x_1, ..., x_n)$ is a vector of mixed strategies, one mixed strategy for each player. We write this set as

$$\boxdot(S) = \times_{i \in I} \Delta_i = \times_{i \in I} \Delta(S_i)$$

- Example: $n = m_1 = m_2 = 2$



- Can you draw a picture of $\boxdot(S)$ when $n = m_2 = 2$ and $m_1 = 3$? (Note that $\boxdot(S)$ then lives in $\mathbb{R}^5$!)

- For each mixed-strategy profile $x \in \boxdot(S)$ and player $i \in I$, let $\tilde{u}_i(x) \in \mathbb{R}$ be the *expected value* of the payoff function u_i when players use their mixed strategies in x :

$$\tilde{u}_i(x) = \mathbb{E}[u_i(s) \mid x] = \sum_{s \in S} \left(\prod_{j=1}^{n} x_j\left(s_j\right) \right) u_i(s)$$

- This completely specifies $\tilde{G} = \langle I, \boxdot(S), \tilde{u} \rangle$, the mixed-strategy extension of any given finite game $G = \langle I, S, u \rangle$

1.2 Existence of Nash equilibrium

- We note that in $\tilde{G} = \langle I, \boxdot(S), \tilde{u} \rangle$ each player's strategy set, Δ_i, is non-empty, convex and compact, and $\tilde{u}(x)$ is continuous in $x \in \boxdot(S)$

- Moreover, for each player i, $\tilde{u}_i(x)$ is linear in the player's own mixed strategy, x_i, for any given strategies used by the other players (when x_i is viewed as a vector in $\mathbb{R}^{m_i}$):

$$\tilde{u}_i(x_i, x_{-i}) = \sum_{h \in S_i} \tilde{u}_i\left(e_i^h, x_{-i}\right) \cdot x_{ih} = a \cdot x_i$$

- ... and all linear functions are quasi-concave!

- Hence, the following is a corollary to last lecture's general existence theorem for Nash equilibrium:

Theorem 1.1 (Nash, 1950) *If G is a finite game, then its mixed-strategy extension $\tilde{G}$ has at least one Nash equilibrium.*

2 Dominance relations

Let G be any finite game with mixed-strategy extension $\tilde{G}$

Definition 2.1 $x_i^* \in \Delta_i$ **strictly dominates** $x_i' \in \Delta_i$ *if*

$\tilde{u}_i(x_i^*, x_{-i}) > \tilde{u}_i(x_i', x_{-i})$ *for all* $x \in \boxdot(S)$

Definition 2.2 $x_i^* \in \Delta_i$ **weakly dominates** $x_i' \in \Delta_i$ *if*

$\tilde{u}_i(x_i^*, x_{-i}) \geq \tilde{u}_i(x_i', x_{-i})$ *for all* $x \in \boxdot(S)$

with $>$ *for some* $x \in \boxdot(S)$

- A strategy that is not weakly dominated is called *undominated*

Definition 2.3 $x_i^* \in \Delta_i$ **is strictly (weakly) dominant** *if it (strictly) weakly dominates all strategies* $x_i' \in \Delta_i$.

- Example: in a Prisoners' dilemma "defect" strictly dominates "cooperate"

Example 2.1

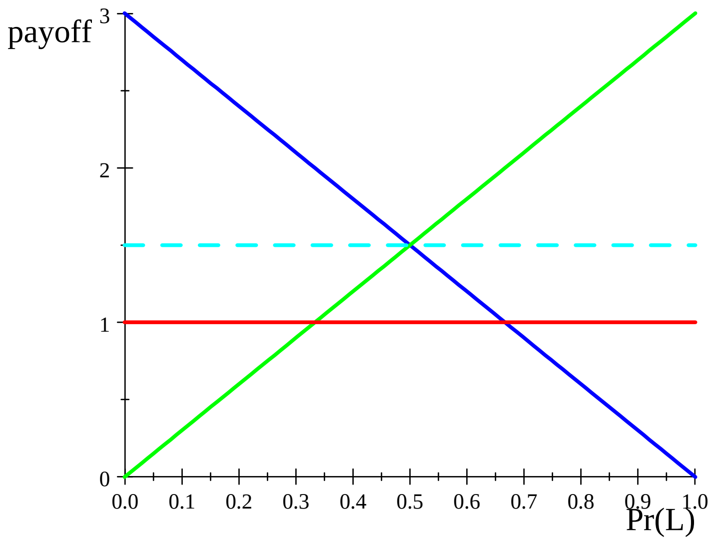
$$\begin{array}{ccc} & A & B \\ A & 9,9 & 0,9 \\ B & 9,0 & 1,1 \end{array}$$

The Nash equilibrium (A,A) gives high payoffs, but both strategies in this equilibrium are weakly dominated!

Which strategy would you use, A or B, or a mixture?

Example 2.2 *Is any of 1's pure strategies strictly dominated in the following game?*

	L	R
T	3, 0	0, 2
M	0, 1	3, 0
B	1, 4	1, 3



- *Iterated elimination* of strictly dominated pure strategies:

$$G = \begin{bmatrix} 3,3 & 1,0 & 6,1 \\ 0,1 & 0,0 & 4,2 \\ 1,6 & 2,4 & 5,5 \end{bmatrix}$$

- One can show, in general, that the order of elimination of *strictly* dominated strategies does not matter for the end result. The remaining non-empty subset of pure strategies, $Q_i \subseteq S_i$, one for each player i, is the same, irrespective of the order used.

Definition 2.4 *A game is* dominance solvable *if* $|Q_i| = 1$ *for each player* i*.*

3 Best replies

- The i:th player's *pure-strategy best-reply correspondence* $\beta_i : \boxdot(S) \rightrightarrows S_i$ is defined by

$$\beta_i(x) = \{h \in S_i : \tilde{u}_i(e_i^h, x_{-i}) \geq \tilde{u}_i(e_i^k, x_{-i})\ \forall k \in S_i\}$$

- Mixed strategies cannot give higher payoffs than pure:

$$\beta_i(x) = \{h \in S_i : \tilde{u}_i(e_i^h, x_{-i}) \geq \tilde{u}_i(x_i', x_{-i})\ \forall x_i' \in \Delta_i\}.$$

- The i:th player's *mixed-strategy best-reply correspondence* $\tilde{\beta}_i$: $\boxdot(S) \rightrightarrows \Delta_i$ is defined by:

$$\begin{aligned} \tilde{\beta}_i(x) &= \{x_i^* \in \Delta_i : \tilde{u}_i(x_i^*, x_{-i}) \geq \tilde{u}_i(x_i', x_{-i})\ \forall x_i' \in \Delta_i\} \\ &= \{x_i^* \in \Delta_i : \textit{supp}(x_i^*) \subseteq \beta_i(x)\} \end{aligned}$$

where *supp*(x_i^*) is the support of x_i^*, that is, the subset $\left\{h \in S_i : x_{ih}^* > 0\right\}$

- Note that $\tilde{\beta}_i(x)$ is a (non-empty) subsimplex!

- The **combined pure** *BR-correspondence* $\beta : \boxdot(S) \rightrightarrows S$:

$$\beta(x) = \times_{i\in I}\beta_i(x)$$

- The **combined mixed** *BR-correspondence* $\tilde{\beta} : \boxdot(S) \rightrightarrows \boxdot(S)$:

$$\tilde{\beta}(x) = \times_{i\in I}\tilde{\beta}_i(x)$$

4 Dominance versus best replies

- Pure best replies are evidently *not* strictly dominated
  - But, if a pure strategy is *not* strictly dominated, is it then a best reply to *some* mixed-strategy profile?

- Pure best replies to *interior* strategy profiles are clearly undominated (why?)
  - But, if a pure strategy is undominated, is it then a best reply to *some* interior mixed-strategy profile?

Proposition 4.1 (Pearce, 1984) *Let G be any finite two-player game and let $s_i \in S_i$ be any strategy for any player $i \in I$.*

(a) s_i is not strictly dominated iff $s_i \in \beta_i(x)$ for some $x \in \boxdot(S)$

(b) s_i is undominated iff $s_i \in \beta_i(x)$ for some $x \in int(\boxdot(S))$

5 Rationalizability

- Let $G = \langle I, S, u \rangle$ be any finite game and assume:

A1 (*Rationality)*: Each player i forms a probabilistic belief $p_j^i \in \Delta\left(S_j\right)$ about every other player j's strategy choice, a belief that does not contradict any information or knowledge that player i has, and player i chooses a (pure or mixed) strategy that maximize his or her expected payoff, assuming statistical independence between different players' strategy choices

A2 (*Common Knowledge*): The game G and the players' rationality (A1) is common knowledge among the players; each player knows G and that (A1) holds for all players, knows that all players know this, and knows that all players know that all players know this etc. *ad infinitum*.

- Question: What is the logical implication of the joint hypothesis $[A1 \wedge A2]$?

- Answer: *Rationalizability*! A concept defined (independently) by David Pearce and Douglas Bernheim in 1984

- Definition based upon the iterated elimination of pure strategies that are not best replies to any mixed-strategy profile

- Both authors showed that every player i has a non-empty subset $R_i \subseteq S_i$ of rationalizable pure strategies

- Recall that $Q_i \subseteq S_i$ is the player's set of pure strategies that survive the iterated elimination of strictly dominated strategies, and write

$$Q = \times_{i \in I} Q_i \text{ and } R = \times_{i \in I} R_i$$

Proposition 5.1 (Pearce, 1984) *Let G be any finite n-player game. Then $R \subseteq Q$, and $R = Q$ if $n = 2$.*

- Reconsider the introductory examples!

6 Nash equilibrium revisited

- Let $G = \langle I, S, u \rangle$ be any finite game with mixed-strategy extension $\tilde{G} = \langle I, \boxdot(S), \tilde{u} \rangle$

- Then $x \in \boxdot(S)$ is a NE of $\tilde{G}$ iff

$$x \in \tilde{\beta}(x)$$

- Equivalently:

$$x_{ih} > 0 \quad \Rightarrow \quad h \in \beta_i(x) \qquad (\forall i \in I, h \in S_i)$$

- All NE are rationalizable:

$$x \in \tilde{\beta}(x) \ \wedge \ x_{ih} > 0 \quad \Rightarrow \quad h \in R_i \qquad (\forall i \in I, h \in S_i)$$

- While a NE strategy cannot be *strictly* dominated, such a strategy may, as noted above, be weakly dominated

Definition 6.1 *A Nash equilibrium* $x = (x_1, .., x_n)$ *is* **undominated** *if no strategy* x_i *is weakly dominated.*

- Practice how to solve for NE in two-player games!

Example 6.1 *Coordination games*

$$\begin{array}{ccc} & A & B \\ A & a_1, b_1 & 0, 0 \\ B & 0, 0 & a_2, b_2 \end{array}$$

for $a_1, b_1, a_2, b_2 > 0$. *Three NE. Solve for the mixed NE! Note how each player's equilibrium randomization depends on the* **other** *player's payoffs (and not at all on the player's own payoffs)! Any completely mixed NE requires indifference:*

$$a_1 x_{21} = a_2 x_{22} \quad \wedge \quad b_1 x_{11} = b_2 x_{12} \quad \Rightarrow \quad x_{11}^* = \frac{b_2}{b_1 + b_2} \text{ etc.}$$

Example 6.2 *Entry-deterrence game: Player 1 has a profitable monopoly in a part of a town, earning 3 million euros per year. Player 2 has a less profitable business in another part of town, earning* 1 *million euros per year. Both are rational and risk-neutral profit maximizer. One day player 2 has an opportunity to move his business into 1's part of town and set up competition there with player 1. Player 1 threatens to then run a price war against 2, resulting in zero profits for both players. If, however, Player 1 would not run a price war after 2's entry, each would earn 2 million euros after 1 entered 1's territory. Should player 2 enter or not? Should player 1, if 2 enters, fight or not? Write this up as a finite normal-form game and find its (infinitely many) Nash equilibria!*

	E	N
F	$0,0$	$3,1$
Y	$2,2$	$3,1$

- We have seen examples of “implausible” Nash equilibria

- Can one discard (some of) those by first principles, by way of using a more refined equilibrium concept?

- Next lecture, we will study two such refinements: *perfection* (Selten, 1975) and *properness* (Myerson, 1978)