# SF2972 GAME THEORY
# Lecture 4: Finite games II

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29 January 2015

- Recall that we have seen examples of “implausible” Nash equilibria (for example in the entry-deterrence game)

- Can one discard (some of) those by first principles, by way of using a more refined equilibrium concept?

- We will now study two such refinements: *perfection* (Selten, 1975) and *properness* (Myerson, 1978)

# 1 Perfect equilibrium

The probably most well-known refinement of Nash equilibrium is that of *"trembling hand" perfection*, due to Selten (1975).

- Selten (1975): "Rationality as the limit of bounded rationality when the bounds on rationality are gradually taken away"

- Players have "trembling hands," and know this

- Imagine that players sometimes, maybe very rarely, make mistakes, when implementing strategies, and are aware of this risk for themselves and for others

- Recall that a strategy profile $x$ is a NE iff

$$h \notin \beta_i(x) \;\Rightarrow\; x_{ih} = 0$$

- The following definition [due to Myerson (1978)] can be shown to be equivalent to Selten's original definition:

**Definition 1.1** *Given $\varepsilon > 0$, a strategy profile $x$ is* $\varepsilon$**-perfect** *if it is interior and*

$$h \notin \beta_i(x) \quad \Rightarrow \; x_{ih} \leq \varepsilon$$

*A* **perfect equilibrium** *is any limit of $\varepsilon$-perfect strategy profiles as $\varepsilon \to 0$.*

- Note that
  - All *strict* NE (that is, where each strategy is the player's *unique* best reply) are perfect
  - All *completely mixed* NE are perfect (Hmm..)

**Theorem 1.1 (Selten, 1975)** *For every finite game $G$, the set of perfect equilibria in $\tilde{G}$ is a nonempty subset of the set of undominated Nash equilibria in $\tilde{G}$.*

**Example 1.1** *Reconsider*

$$\begin{array}{ccc} & A & B \\ A & 9,9 & 0,9 \\ B & 9,0 & 1,1 \end{array}$$

*The only perfect equilibrium is the undominated NE (B,B)*

**Example 1.2** *Reconsider the entry-deterrence game:*

$$\begin{array}{ccc} & E & N \\ F & 0,0 & 3,1 \\ Y & 2,2 & 3,1 \end{array}$$

*The only perfect equilibrium is the only "plausible" NE: (E,Y), that is, entry followed by yielding. This is a strict and hence perfect equilibrium. Why is (F,N) not perfect?*

**Proposition 1.2 (van Damme, 1987)** *In two-player games all undominated Nash equilibria are perfect.*

- Perfection is based on first principles and does a great job in many, if not most games!

- However, it only requires robustness against *some* trembles, so it is sufficient to find *any* sequence of smaller and smaller trembles, whether or not they are plausible

- Myerson (1978) pointed out that this has the implication that a non-perfect NE may become perfect if one adds a strictly dominated strategy to the game - an arguably undesirable property of a solution concept

**Example 1.3** *Add a "dumb" strategy to the entry-deterrence game:*

$$\begin{array}{cccc} & E & N & D \\ F & 0,0 & 3,1 & 2,-2 \\ Y & 2,2 & 3,1 & 1,-1 \end{array}$$

*In the original game* $(F, N)$ *was not perfect, but now it is! For the monopolist F is better than Y against D. For all small* $\varepsilon > 0$, *here is an* $\varepsilon$*-perfect strategy profile* $x^{\varepsilon} \longrightarrow (F, N)$ :

$$\begin{cases} x^{\varepsilon}_{1F} = 1 - \varepsilon \ , \quad x^{\varepsilon}_{1Y} = \varepsilon \\ x^{\varepsilon}_{2N} = 1 - 4\varepsilon/3 \ , \quad x^{\varepsilon}_{2D} = \varepsilon \ , \quad x^{\varepsilon}_{2E} = \varepsilon/3 \end{cases}$$

$$\begin{cases} \tilde{u}_1 (F, x^{\varepsilon}) = 3\,(1 - 4\varepsilon/3) + 2\varepsilon \\ \tilde{u}_1 (Y, x^{\varepsilon}) = 2\varepsilon/3 + 3\,(1 - 4\varepsilon/3) + \varepsilon \end{cases}$$

$$\begin{cases} \tilde{u}_2 (E, x^{\varepsilon}) = 2\varepsilon \\ \tilde{u}_2 (N, x^{\varepsilon}) = 1 \qquad (\textit{and } 2\varepsilon < 1 \textit{if } \varepsilon < 1/2) \\ \tilde{u}_2 (D, x^{\varepsilon}) < 0 \end{cases}$$

# 2 Proper equilibria

- Myerson (1978) suggested the following further refinement of NE, based on the observation that people seem to be less likely to make more costly mistakes, and others seem to be aware of that

**Definition 2.1 (Myerson, 1978)** *Given $\varepsilon > 0$, a strategy profile $x$ is* $\varepsilon$-**proper** *if it is interior and*

$$\tilde{u}_i\left(e_i^h, x_{-i}\right) < \tilde{u}_i\left(e_i^k, x_{-i}\right) \quad \Rightarrow \quad x_{ih} \leq \varepsilon \cdot x_{ik}$$

*A* **proper equilibrium** *is any limit of $\varepsilon$-proper strategy profiles as $\varepsilon \to 0$.*

- Note that every $\varepsilon$-proper strategy profile is $\varepsilon$-perfect

- Note also that (as with perfection):

- All *completely mixed* NE are proper

- All *strict* NE are proper (can be shown)

- Is it to ask for too much, to ask for properness, so that some games have no proper equilibrium?

**Proposition 2.1 (Myerson, 1978)** *The set of proper equilibria is a non-empty subset of the set of perfect equilibria.*

**Example 2.1** *The augmented entry-deterrence game*

| | $E$ | $N$ | $D$ |
|---|---|---|---|
| $F$ | $0,0$ | $3,1$ | $2,-2$ |
| $Y$ | $2,2$ | $3,1$ | $1,-1$ |

*While we saw earlier that $(F,N)$ is perfect (though not credible). It is not proper, because when play is close to $(F,N)$, the mistake D is more costly to player 1 than the mistake E: Let*

$$x^{\varepsilon}_{1F} = 1-\varepsilon\ , \quad x^{\varepsilon}_{1Y} = \varepsilon$$

*for some small $\varepsilon > 0$. Then*

$$\begin{cases} \tilde{u}_2\left(E, x^{\varepsilon}\right) = 2\varepsilon \\ \tilde{u}_2\left(N, x^{\varepsilon}\right) = 1 \\ \tilde{u}_2\left(D, x^{\varepsilon}\right) < 0 \end{cases}$$

*so if $x^{\varepsilon}_{2E} \leq \varepsilon$ then $\varepsilon-$properness requires that $x^{\varepsilon}_{2D} \leq \varepsilon^2$. For sufficiently small $\varepsilon > 0$ this makes $Y$ a better reply to $x^{\varepsilon}_2$ violating the hypothesis that $x^{\varepsilon}_{1F} = 1-\varepsilon$ and $x^{\varepsilon}_{1Y} = \varepsilon$, so $(F,N)$ is not proper.*

*The only proper equilibrium is the arguably more plausible $(Y,E)$.*

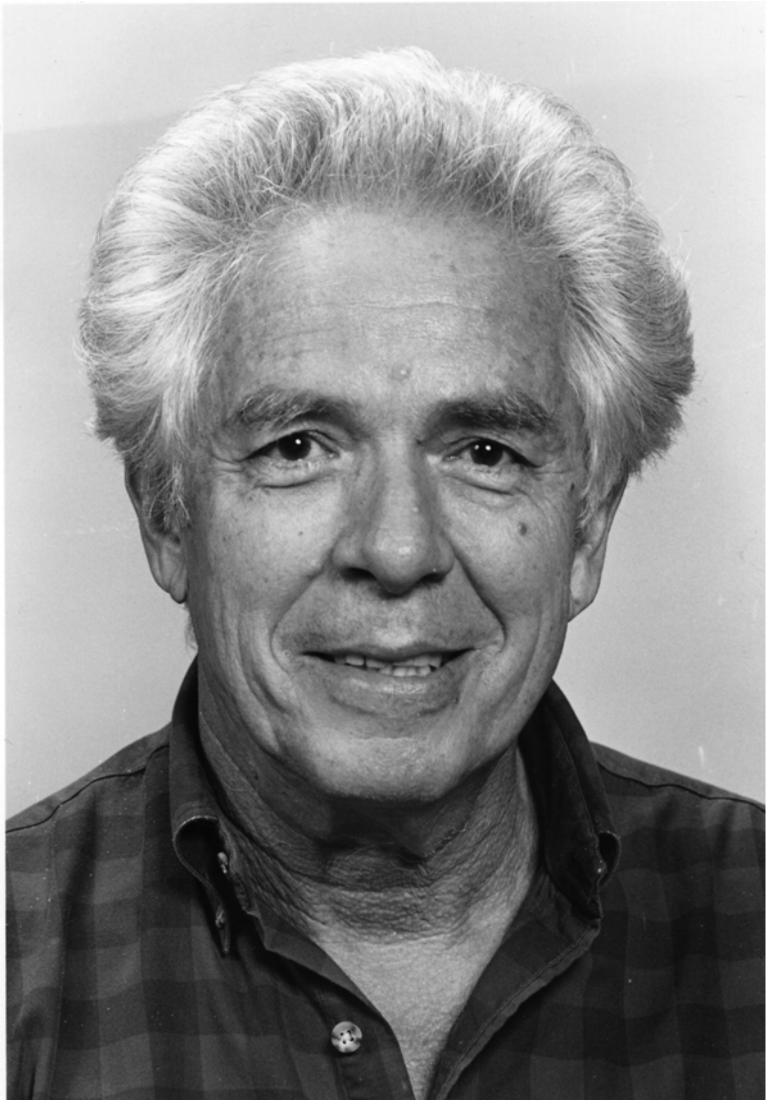
- In addition, it has later been shown that properness has an amazing implication for extensive-form analysis! ... The topic for the next few lectures, given by Mark Voorneveld.

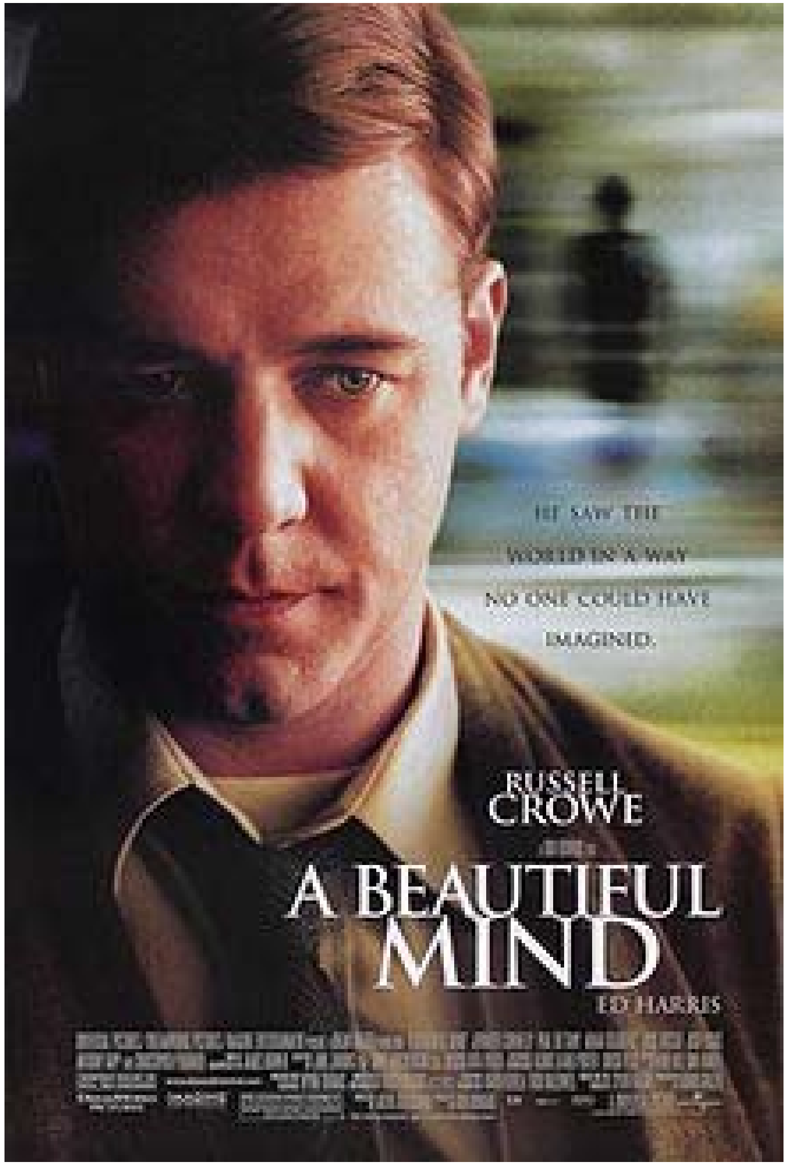
# 3 Summary of our normal-form analysis

See picture in separate PDF-file!

# 4 An informal introduction to extensive-form analysis

- The extensive-form representation of a strategic interaction specifies in detail who moves when and what information is then available
- Like the normal form, it is a mathematical structure, but more complex
- It takes the form of a multi-person decision tree. Its format was developed by a class-mate of John Nash at Princeton, Harold Kuhn (1925-2014, also known for the Kuhn-Tucker Theorem in optimization)




HE SAW THE
WORLD IN A WAY
NO ONE COULD HAVE
IMAGINED.
RUSSELL
CROWE
A BEAUTIFUL
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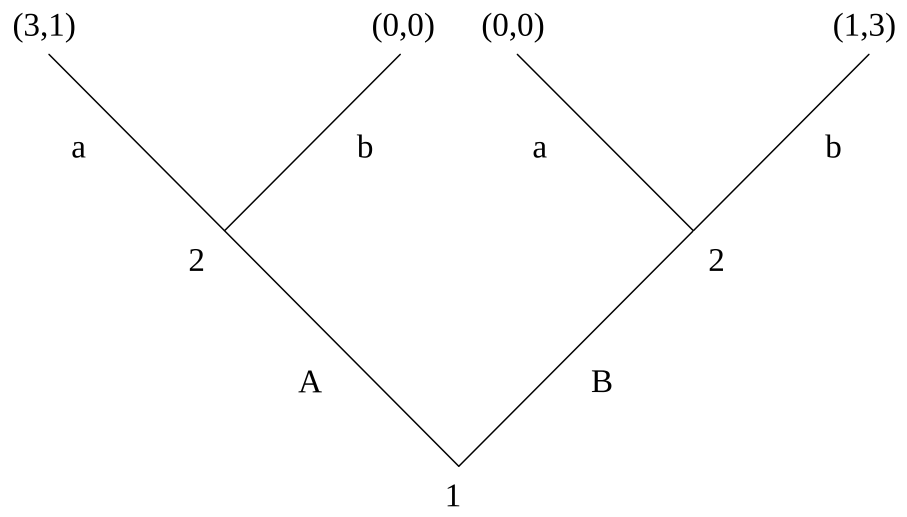
- If you are one of the players in a two-player game, would you like to first be informed about the other player's move, before you make your move, or would you instead prefer that the other player is informed about your move before making his/her?



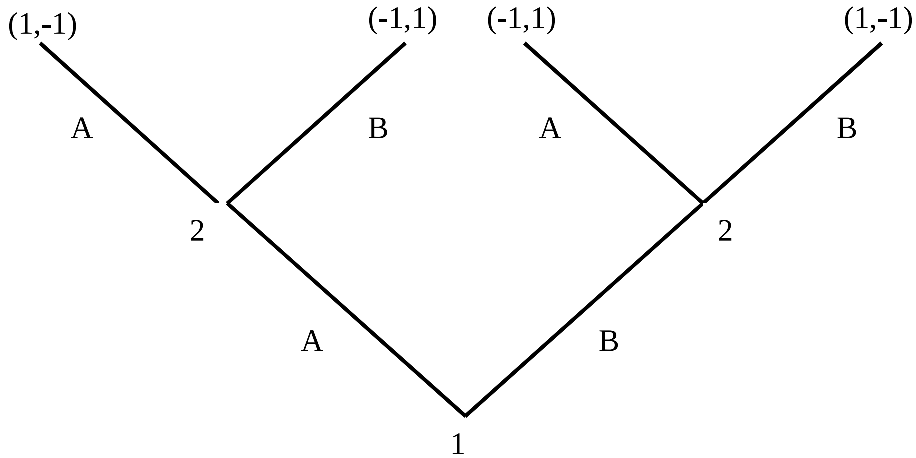

- Player 2 is informed about player 1's move, before 2 makes her move
- A game of perfect information
- Better to be uninformed, to be the first mover (a first-mover advantage)

- Are there games in which it is better to be informed, that is, with a *second*-mover advantage?

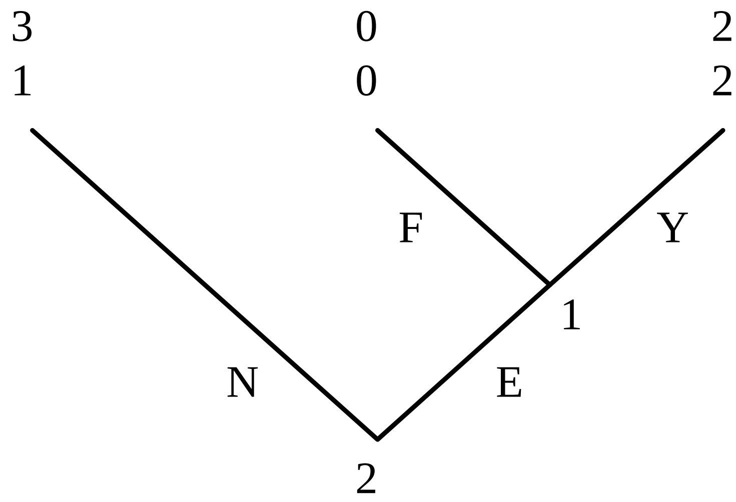
- A fox (or Moriarty, player 1) and a rabbit (or Sherlock Holmes, player 2), each choosing between two locations, A and B

- If you were the rabbit (fox), would you like to choose first or second?

- If the fox chooses first:



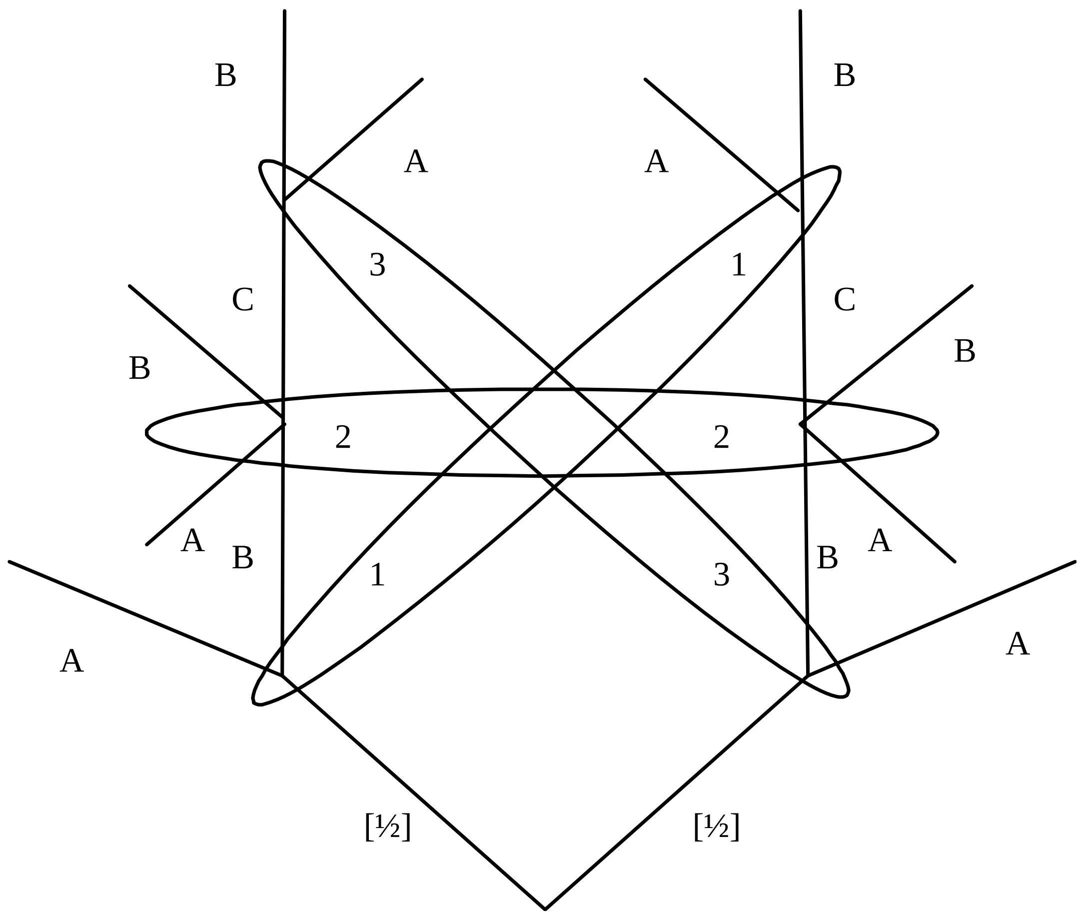
- An extensive-form represenation of the *entry-deterrence* game




- Its normal form:

| | $E$ | $N$ |
|---|---|---|
| $F$ | $0, 0$ | $3, 1$ |
| $Y$ | $2, 2$ | $3, 1$ |

- And there are much more complex extensive-form games, with many players, some moves in sequence, some simultaneous, imperfect information that changes as play proceeds etc.



- The indicated amazing property of proper equilibrium is that every such equilibrium induces a *sequential equilibrium* in every extensive-form game with the given normal form (van Damme, 1984)!

- Isn't that beautiful?

THE END