GAME THEORY EXERCISES
for Seminar 1

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KTH February 4, 2015

1. [Cournot oligopoly] Consider n firms competing in a product market with demand $D(p) = \max\{0, 100 - p\}$. In stage 1 of the interaction, each firm i selects its output level $q_i \in [0, 100]$, without observing other firms' outputs. Let $Q = q_1 + ... + q_n$. In stage 2, the market price is determined by the market-clearing condition $D(p) = Q$. Suppose that the profit to firm i is $\pi_i = (p - c_i) q_i$, for some (firm-specific unit production costs) $c_i < 50$. Suppose that each firm strives to maximize its profit.

   (a) Let $n = 2$ and suppose that both firms' managers are rational and that their rationality and the game (including their costs) is common knowledge between them. Show that this uniquely determines their output levels, and that this coincides with the unique Nash equilibrium of the game. Calculate the equilibrium price and each firm's equilibrium profit.

   (b) Let n be any positive integer, and suppose that $c_1 = c_2 = ... = c_n = c$ for some c. Show that there exists a unique Nash equilibrium and that this is symmetric (all firms produce the same quantity). Calculate the market price and the equilibrium profits, and study how these depend on the number n of firms and on the unit production cost c.

   (c) In (b), let $n \to \infty$ and study (and explain) the limit equilibrium price and profits.

2. Consider the two-player normal-form game

	L	RA	RB
LA	$1, 1$	$2, -2$	$-2, 2$
LB	$1, 1$	$-2, 2$	$2, -2$
R	$0, 0$	$1, 1$	$1, 1$

   (a) Find all rationalizable pure strategies.

   (b) Find all Nash equilibria (in pure and mixed strategies).

   (c) Find all perfect equilibria (in pure and mixed strategies).

(d) Find all proper equilibria (in pure and mixed strategies).

3. [Partnership game] There are $n \geq 1$ partners who together own a firm. Each partner i chooses an effort level $x_i \geq 0$, resulting in total profit $g(y)$ for their firm, where $y = x_1 + .. + x_n$ is their aggregate effort. The profit function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous with $g(0) = 0$, and it is twice differentiable on $\mathbb{R}_{++}$ with $g' > 0$, and $g'' \leq 0$. The firm's profit is shared equally by the partners, and each partner's effort gives him or her (quadratic) disutility. The resulting utility level for each partner i is

$$u_i(x_1, .., x_n) = \frac{1}{n} \cdot g\left(x_1 + .. + x_n\right) - x_i^2/2$$

where $(x_1, .., x_n)$ is the effort profile.

(a) Suppose each partner i has to decide his or her effort x_i without observing the others' efforts. Show that the game has exactly one Nash equilibrium, and show that all partners make the same effort, x^*, in equilibrium. Is the individual equilibrium effort x^* increasing or decreasing in n, or is it independent of n? Is the aggregate equilibrium effort, $y^* = nx^*$, increasing or decreasing in n, or is it independent of n?

(b) Suppose that the partners can pre-commit to a common effort level, $x \geq 0$, the same for all. Let $\hat{x}$ be the common effort level that maximizes the sum of the partners' utilities. Characterize $\hat{x}$ in terms of an equation, and compare this level with the equilibrium effort x^* in (a), for $n = 1, 2, ...$ Are the partners better off now than in the equilibrium in (a)? How does this depend on n? Explain!

(c) Now consider the special case of a linear profit function, $g\left(y\right) = y$. Find explicit solutions for (a)-(b) and discuss how and why these solutions depend on the number $n \geq 1$of partners in the firm.

4. Consider the two-player normal-form game with payoff bimatrix

	L	m	R
T	$5,5$	$3,0$	$0,2$
M	$5,1$	$2,1$	$1,0$
B	$0,0$	$2,5$	$4,2$

(a) Find all rationalizable pure strategies.

(b) Find all pure-strategy Nash equilibria.

(c) Find all pure-strategy perfect equilibria.