SOLUTIONS TO GAME THEORY EXERCISES in Seminar 1

JÖRGEN WEIBULL

KTH February 4, 2015

1. [Cournot oligopoly] Consider n firms competing in a product market with demand $D(p) = \max\{0, 100 - p\}$. In stage 1 of the interaction, each firm i selects its output level $q_i \in [0, 100]$, without observing other firms' outputs. Let $Q = q_1 + ... + q_n$. In stage 2, the market price is determined by the market-clearing condition $D(p) = Q$. Suppose that the profit to firm i is $\pi_i = (p - c_i) q_i$, for some (firm-specific unit production costs) $c_i < 50$. Suppose that each firm strives to maximize its profit.

   (a) Let $n = 2$ and suppose that both firms' managers are rational and that their rationality and the game (including their costs) is common knowledge between them. Show that this uniquely determines their output levels, and that this coincides with the unique Nash equilibrium of the game. Calculate the equilibrium price and each firm's equilibrium profit.

   Solution: (See also Peters 6.2.1) The payoff function for each firm i is

$$u_i(q_1, q_2) = [\max\{0, 100 - q_1 + q_2\} - c_i] q_i$$

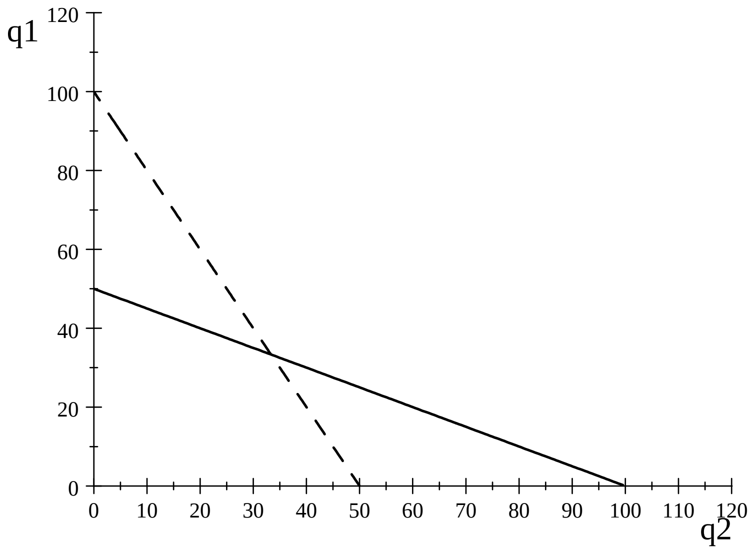
   Use the F.O.C. to find firm 1's best reply to any given q_2:

$$\beta_1(q_2) = \frac{1}{2}\max\{0, 100 - q_2 - c_1\}$$

   Likewise for firm 2:

$$\beta_2(q_1) = \frac{1}{2}\max\{0, 100 - q_1 - c_2\}$$

   Diagram drawn for for $c_1 = c_2 = 0$:



Rationalizability: iteratively remove pure strategies q_i that are not BR. This results in an infinite sequence of shrinking intervals, just as in class in Lecture 1. The only remaing pure strategy for each player is the one that corresponds to the intersection of the two straight lines, the NE.

NE: The intersection of the two BR-curves gives the unique NE. Its aggregate output and price:

$$Q^* = \frac{2}{3}(100 - \bar{c}) \qquad p^* = 100 - Q^* = \frac{1}{3}100 + \frac{2}{3}\bar{c}$$

where $\bar{c} = (c_1 + c_2)/2$. From this one can calculate q_1^*, q_2^*, π_1^* and π_2^*.

(b) Let n be any positive integer, and suppose that $c_1 = c_2 = ... = c_n = c$ for some c. Show that there exists a unique Nash equilibrium and that this is symmetric (all firms produce the same quantity). Calculate the market price and the equilibrium profits, and study how these depend on the number n of firms and on the unit production cost c.

Solution: Just as in (a), the best reply of any firm i to the aggregate output Q_{-i} of other firms is

$$\beta_i(Q_{-i}) = \frac{1}{2}\max\{0, 100 - Q_{-i} - c\}$$

Nash equilibrium requires that $q_i = \beta_i(Q_{-i})$ for all i. If $c \geq 100$, then we obtain $q_i^* = 0$ for all firms i. (No aggregate output since then c is too high to make any positive output profitable.) Suppose $c < 100$. Then

$Q^* > 0$. (If $Q^* = 0$, every firm could profitably deviate by producing something.) Suppose, without loss of generality, that $q_i^* > 0$ for all $i \leq k$, and that $q_i^* = 0$ for all $i > k$. If $k < n$, then some firms have positive output, so $Q_{-n} > 0$ and they make positive profits (since otherwise one of them could reduce its positive profits somewhat), so $\beta_n (Q_{-n}) > 0$, a contradiction. Hence $k = n$, that is, all firms have positive output and thus in any NE:

$$q_i = \frac{1}{2}(100 - Q_{-i} - c) \quad \forall i$$

Equivalently,

$$q_i = 100 - Q - c \quad \forall i$$

so all q_i are the same. Hence, every NE is symmetric:

$$q_i = q^* = 100 - Q^* - c \quad \forall i$$

Solving for $Q^* = nq^*$ and p^*:

$$Q^* = \frac{n}{n+1}(100 - c) \qquad p^* = \frac{1}{n+1}100 + \frac{n}{n+1}c$$

(c) In (b), let $n \to \infty$ and study (and explain) the limit equilibrium price and profits.

Solution: p^* increases with n and approaches c as $n \to \infty$. Each firm's equilibrium profit thus decreases with n and approaches zero as $n \to \infty$. The limit case is the same as what is called perfect competition in economics.

2. Consider the two-player normal-form game

	L	RA	RB
LA	$1,1$	$2,-2$	$-2,2$
LB	$1,1$	$-2,2$	$2,-2$
R	$0,0$	$1,1$	$1,1$

(a) Find all rationalizable pure strategies.

Solution: In finite two-player games, a pure strategy is rationalizable iff it remains after the the iterated elimination of all strictly dominated pure strategies. No strategy in this game is strictly dominated. Hence, all strategies are rationalizable.

(b) Find all Nash equilibria (in pure and mixed strategies).

Solution: Inspection of the payoff bi-matrix shows that this game has no NE in pure strategies. We thus have to look for mixed NE. This game has infinitely many NE (writing pure strategy names within square brackets):
(i) All strategy profiles x^* in which 2 plays L and $x_1^* = p\cdot[LA]+(1-p)\cdot[LB]$ for $1/4 \leq p \leq 3/4$.
(ii) All strategy profiles x^* in which 1 plays R and $x_2^* = q\cdot[RA]+(1-q)\cdot[RB]$ for $1/4 \leq q \leq 3/4$
(iii) The completely mixed NE

$$x^M = (\frac{1}{2}\cdot[R] + \frac{1}{4}\cdot[LA] + \frac{1}{4}\cdot[LB])\ ,\ \frac{1}{2}\cdot[L] + \frac{1}{4}\cdot[RA] + \frac{1}{4}\cdot[RB]).$$

(c) Find all perfect equilibria (in pure and mixed strategies).

Solution: In finite two-player games, a NE is perfect iff it is undominated. Since there are no weakly dominated strategies in the game, all NE are perfect.

(d) Find all proper equilibria (in pure and mixed strategies).

Solution: There are 3 proper equilibria (one in the middle of each of the sets above):

$$x^L = (\frac{1}{2}[LA] + \frac{1}{2}[LB], [L]),\ x^R = ([R], \frac{1}{2}[RA] + \frac{1}{2}[RB]) \text{ and } x^M.$$

That x^M is proper follows from the fact that it is completely mixed, and all completely mixed NE are proper. For x^L, consider all NE in which player 2 plays L. Suppose 2, in a completely mixed strategy profile close to x^L attaches more probabilities to RA than to RB. Then ε-properness requires player 1 to move put much more probability on LA than on LB. But then RA is a worse reply for 2 than RB, so 2 should have put more probability on RB, a contradiction. Hence, 2 needs to put equal probability on RA and RB. If 1 places equal probability on LA and LB, then this makes up an ε-proper strategy profile, and as $\varepsilon \to 0$, it approaches x^L. Likewise for x^R.

3. [Partnership game] There are $n \geq 1$ partners who together own a firm. Each partner i chooses an effort level $x_i \geq 0$, resulting in total profit $g(y)$ for their firm, where $y = x_1 + .. + x_n$ is their aggregate effort. The profit function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous with $g(0) = 0$, and it is twice differentiable on $\mathbb{R}_{++}$ with $g' > 0$, and $g'' \leq 0$. The firm's profit is shared equally by the partners,

and each partner's effort gives him or her (quadratic) disutility. The resulting utility level for each partner i is

$$u_i(x_1, .., x_n) = \frac{1}{n} \cdot g\,(x_1 + .. + x_n) - x_i^2/2$$

where $(x_1, .., x_n)$ is the effort profile.

(a) Suppose each partner i has to decide his or her effort x_i without observing the others' efforts. Show that the game has exactly one Nash equilibrium, and show that all partners make the same effort, x^*, in equilibrium. Is the individual equilibrium effort x^* increasing or decreasing in n, or is it independent of n? Is the aggregate equilibrium effort, $y^* = nx^*$, increasing or decreasing in n, or is it independent of n?

Solution: For any strategy profile and partner i we have

$$\frac{\partial u_i(x_1, .., x_n)}{\partial x_i} = \frac{1}{n} \cdot g'\,(y) - x_i$$

Since $g' > 0$, all partners necessarily make positive efforts in any NE. Moreover, their efforts are necessarily the same: the necessary F.O.C. implies

$$x_i = \frac{1}{n} \cdot g'\,(y) \quad \forall i$$

Summation over the partners implies that

$$y^* = g'\,(y^*)$$

in NE. This (fixed-point) equation has a unique solution since the right-hand side is continuous and non-increasging ($g'' \leq 0$). Moreover, since each payoff function in fact is strictly concave, the unique NE is $x_i = x^* = y^*/n$ for all partners i. Clearly aggregate NE effort, y^* is independent of n. Thus, individual NE effort x^* is decreasing in n.

(b) Suppose that the partners can pre-commit to a common effort level, $x \geq 0$, the same for all. Let $\hat{x}$ be the common effort level that maximizes the sum of the partners' utilities. Characterize $\hat{x}$ in terms of an equation, and compare this level with the equilibrium effort x^* in (a), for $n = 1, 2, ...$ Are the partners better off now than in the equilibrium in (a)? How does this depend on n? Explain!

Solution: The sum of all partner's utilities, when they each exert effort x is simply

$$W\,(x) = g\,(nx) - nx^2/2$$

The necessary and sufficient F.O.C. for maximization of this (welfare) function is clearly $ng(nx) = nx$ or $y = ng(y)$, again an equation that has a unique solution, $\hat{y}$. Clearly $\hat{y} > y^*$ iff $n > 1$. All partners contribute more and are achieve higher utility than in Nash equilibrium, for all $n > 1$.

(c) Now consider the special case of a linear profit function, $g(y) = y$. Find explicit solutions for (a)-(b) and discuss how and why these solutions depend on the number $n \geq 1$of partners in the firm.

Solution: Now $g'(y) = 1$ for all y, so $y^* = 1$ (and thus $x^* = 1/n$) while $\hat{y} = n$ (and thus $\hat{x} = 1$, irrespective of n).

4. Consider the two-player normal-form game with payoff bimatrix

	L	m	R
T	$5,5$	$3,0$	$0,2$
M	$5,1$	$2,1$	$1,0$
B	$0,0$	$2,5$	$4,2$

(a) Find all rationalizable pure strategies.

Solution: In two-player games a pure strategy is rationalizable iff it survives the iterated elimination of strictly dominated pure strategies. Strategy R is strictly dominated by the mixed strategy $x_2 = 0.5[L] + 0.5[m]$. After pure strategy R has been eliminated, strategy B is strictly dominated by T. After also B has been eliminated, no more pure strategies can be eliminated. Hence, the rationalizable pure strategies for player 1 are T and M, and for player 2 they are L and m.

(b) Find all pure-strategy Nash equilibria.

Solution: Suffient to look among rationalizable strategies. The pure-strategy NE are (T,L) and (M,L).

(c) Find all pure-strategy perfect equilibria.

Solution: In two-player games, a NE is perfect iff it is undominated. Neither T nor L is weakly dominated, so (T,L) is perfect. Neither M nor L is weakly dominated (in the full game), so also (M,L) is perfect.