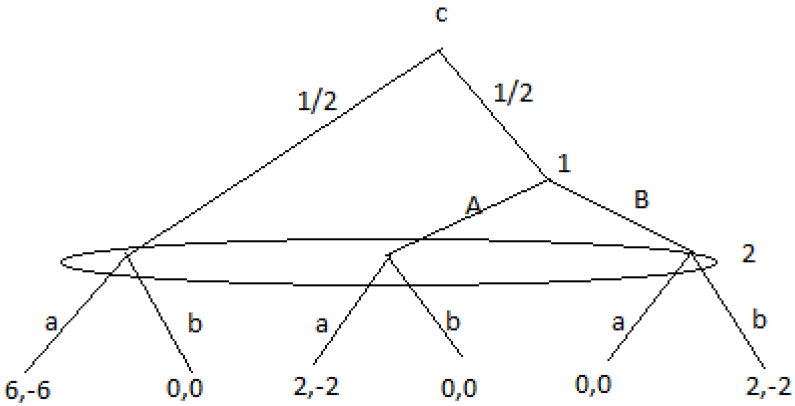
SF2972: Game theory

Mark Voorneveld, mark.voorneveld@hhs.se

February 2017

- Topic: extensive form games.
- Purpose: (1) explicitly model situations in which players move sequentially and their information; (2) formulate appropriate equilibrium notions.
- Textbook (Peters): chapters 4, 5, 14.
- Notes (Weibull): chapter 3
- Reading guide at end of these slides.

Drawing a game tree is usually the most informative way to represent an extensive form game. Here is one with an initial (c)hance move:



Extensive form game: formal definition

- A (directed, rooted) tree; i.e. it has a well-defined initial node.
- Nodes can be of three types:
  1. chance nodes: where chance/nature chooses a branch according to a given/known probability distribution;
     Let τ assign to each chance node a prob distr over feasible branches.
  2. decision nodes: where a player chooses a branch;
  3. end nodes: where there are no more decisions to be made and each player i gets a payoff/utility given by a utility function u_i.
- A function P assigns to each decision node a player i in player set N who gets to decide there.
- Decision nodes $P^{-1}(i)$ of player i are partitioned into information sets.
  Nodes in an information set of player i are 'indistinguishable' to player i; this requires, for instance, the same actions in each decision node of the information set.
- If h is an information set of player i, write $P(h) = i$ and let $A(h)$ be the feasible actions in info set h.

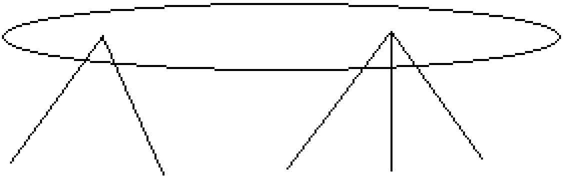
Extensive form game: formal definition

In summary, an extensive form game is a tuple $(T, N, P, \mathcal{H}, \mathcal{A}, \tau, (u_i)_{i \in N})$ with

- T the tree,
- N the set of players,
- P a mapping from decision nodes to the players,
- $\mathcal{H}$ the collection of information sets,
- $\mathcal{A}$ the collection of actions in these information sets,
- τ the prob distr over actions in nature's/chance's nodes,
- $(u_i)_{i \in N}$ the players' (vNM) utility functions.

Notational conventions

- p. 253: "Clearly, this formal notation is quite cumbersome and we try to avoid its use as much as possible. It is only needed to give precise definitions and proofs." *Draw tree!*
- Nodes in same information set: dotted lines between them (Peters' book) or enclosed in an oval (my drawings).
- Since nodes in an information set are indistinguishable, information sets like



are not allowed: since there are two branches in the left node and three in the right, they are easily distinguishable.

We call an extensive form game *finite* if it has finitely many nodes.
An extensive form game has

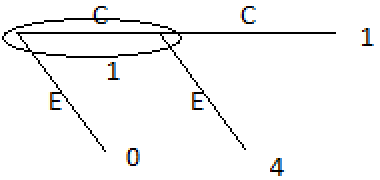
- *perfect information* if each information set consists of only one node.
- *perfect recall* if each player recalls exactly what he did in the past.
  Formally: on the path from the initial node to a decision node x of player i, list in consecutive order which information sets of i were encountered and what i did there. Call this list the experience $X_i(x)$ of i in node x. The game has perfect recall if nodes in the same information set have the same experience.
- otherwise, the game has imperfect information/recall.

Convention: we often characterize nodes in the tree by describing the sequence of actions that leads to them. For instance:

- the initial node of the tree is denoted by $\emptyset$;
- node (a_1, a_2, a_3) is reached after three steps/branches/actions: first a_1, then a_2, then a_3.

Imperfect recall: absentminded driver

Two crossings on your way home. You need to (C)ontinue on the first, (E)xit on the second. But you don't recall *whether* you already passed a crossing.

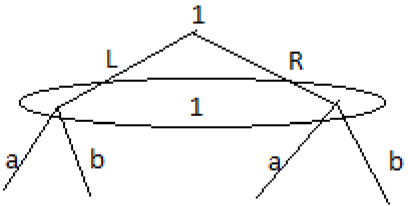


Only one information set, $\{\emptyset, C\}$, but with different experiences:

- in the first node: $X_1(\emptyset) = (\{\emptyset, C\})$
- in the second node:
  $$X_1(C) = (\underbrace{\{\emptyset, C\}}_{\text{1's first info set}}, \underbrace{C}_{\text{choice there}}, \underbrace{\{\emptyset, C\}}_{\text{resulting info set}})$$
- $X_1(\emptyset) \neq X_1(C)$: imperfect recall!

Second example of imperfect recall

Player 1 forgets the initial choice:

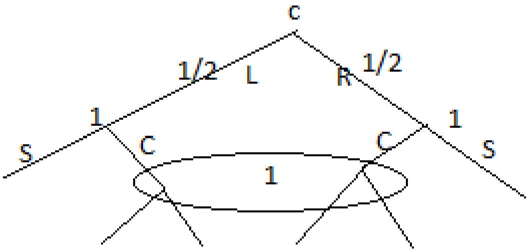


Different experiences in the two nodes of information set $\{L, R\}$:

- in the left node: $X_1(L) = (\underbrace{\emptyset}_{\text{initial node}}, \underbrace{L}_{\text{choice there}}, \underbrace{\{L, R\}}_{\text{resulting info set}})$
- in the right node: $X_1(R) = (\emptyset, R, \{L, R\})$.
- $X_1(L) \neq X_1(R)$: imperfect recall!

Third example of imperfect recall

Player 1 knew the chance move, but forgot it:



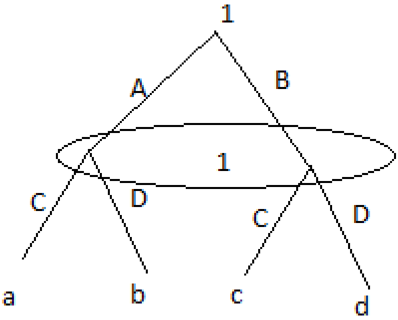
Different experiences in the two nodes of information set $\{(L, C), (R, C)\}$:

- in the left node:
  $$X_1((L, C)) = (\underbrace{\{L\}}_{\text{1's first info set}}, \underbrace{C}_{\text{choice there}}, \underbrace{\{(L, C), (R, C)\}}_{\text{resulting info set}})$$
- in the right node: $X_1((R, C)) = (\{R\}, C, \{(L, C), (R, C)\})$.
- $X_1((L, C)) \neq X_1((R, C))$: imperfect recall!

Pure, mixed, and behavioral strategies

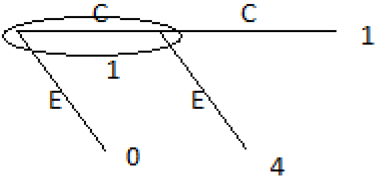
- A *pure strategy* of player i is a function s_i that assigns to each information set h of player i a feasible action $s_i(h) \in A(h)$.
- A *mixed strategy* of player i is a probability distribution σ_i over i's pure strategies.
  $\sigma_i(s_i) \in [0, 1]$ is the prob assigned to pure strategy s_i.
  'Global randomization' at the beginning of the game.
- A *behavioral strategy* of player i is a function b_i that assigns to each information set h of player i a probability distribution over the feasible actions $A(h)$.
  $b_i(h)(a)$ is the prob of action $a \in A(h)$.
  'Local randomization' as play proceeds.

Let us consider the difference between these three kinds of strategies in a few examples.



- Imperfect recall; 4 outcomes with payoffs a, b, c, and d.
- Four pure strategies, abbreviated AC, AD, BC, BD.
- Mixed strategies: probability distributions over the 4 pure strategies. A vector $(p_{AC}, p_{AD}, p_{BC}, p_{BD})$ of nonnegative numbers, adding up to one, with p_x the probability assigned to pure strategy $x \in \{AC, AD, BC, BD\}$.

- Behavioral strategies assign to each information set a probability distribution over the available actions. Since pl. 1 has 2 information sets, each with 2 actions, it is summarized by a pair $(p, q) \in [0, 1] \times [0, 1]$, where $p \in [0, 1]$ is the probability assigned to action A in the initial node (and $1 - p$ to B) and q is the probability assigned to action C in information set $\{A, B\}$ (and $1 - q$ to D).
- Mixed strategy $(1/2, 0, 0, 1/2)$ assigns probability $1/2$ to each of the outcomes a and d. There is no such behavioral strategy:
  - reaching a with positive probability requires that $p, q > 0$;
  - reaching d with positive probability requires $p, q < 1$;
  - hence also b and c are reached with positive probability.



- Pure strategies: C with payoff 1 and E with payoff 0.
- Mixed: let $p \in [0,1]$ be the prob of choosing pure strategy C and $1-p$ the prob of pure strategy E. Expected payoff: p.
- Behavioral: let $q \in [0,1]$ be the prob of choosing action C in the info set and $1-q$ the prob of choosing E in the info set. Expected payoff:

$$0 \cdot (1-q) + 4 \cdot q(1-q) + 1 \cdot q^2 = q(4-3q).$$

- No behavioral strategy is outcome-equivalent with $p = 1/2$ (why?)
- No mixed strategy is outcome-equivalent with $q = 1/2$ (why?)

Conclude: under imperfect recall, mixed and behavioral strategies might generate different probability distributions over end nodes. Perfect recall helps to rule this out. We need a few definitions:

Each profile $b = (b_i)_{i \in N}$ of *behavioral strategies* induces an outcome $O(b)$, a probability distribution over end nodes.

How to compute $O(b)$ in finite games?

The probability of reaching end node $x = (a_1, \ldots, a_k)$, described by the sequence of actions/branches leading to it, is simply the product of the probabilities of each separate branch:

$$\prod_{\ell=0}^{k-1} b_{P(a_1,\ldots,a_\ell)}(a_1, \ldots, a_\ell)(a_{\ell+1}).$$

Likewise, each profile $\sigma = (\sigma_i)_{i \in N}$ of *mixed strategies* induces an outcome $O(\sigma)$, a probability distribution over end nodes.
How to compute $O(\sigma)$ in finite games?

- Let $x = (a_1, \ldots, a_k)$ be a node, described by the sequence of actions/branches in the game tree leading to it.
- Pure strategy s_i of player i is *consistent with* x if i chooses the actions described by x: for each initial segment $(a_1, \ldots, a_\ell)$ with $\ell < k$ and $P(a_1, \ldots, a_\ell) = i$:
$$s_i(a_1, \ldots, a_\ell) = a_{\ell+1}.$$
- The prob of i choosing a pure strategy s_i consistent with x is
$$\pi_i(x) = \sum \sigma_i(s_i),$$
with summation over the s_i consistent with x.
- Similar for nature, whose behavior is given by function τ.
- The probability of reaching end node x is
$$\prod_{i \in N \cup \{c\}} \pi_i(x).$$

A mixed strategy σ_i and a behavioral strategy b_i of player i are *outcome-equivalent* if — given the pure strategies of the remaining players — they give rise to the same outcome:

$$\text{for all } s_{-i}: \quad O(\sigma_i, s_{-i}) = O(b_i, s_{-i}).$$

Theorem (Kuhn, outcome equivalence under perfect recall)

In a finite extensive form game with perfect recall:

(a) *each behavioral strategy has an outcome-equivalent mixed strategy,*

(b) *each mixed strategy has an outcome-equivalent behavioral strategy.*

Proof sketch:

(a) Given beh. str. b_i, assign to pure strategy s_i the probability

$$\sigma_i(s_i) = \prod_h b_i(h)(s_i(h)),$$

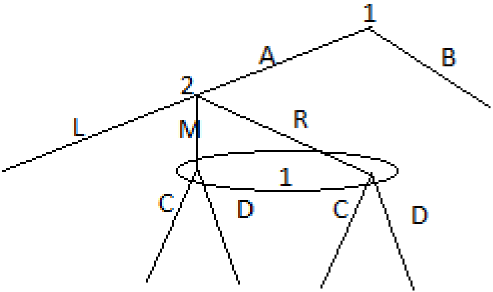
with the product taken over all info sets h of pl i.
Intuition: s_i selects action $s_i(h)$ in information set h. How likely is that?

(b) Given mixed str. σ_i. Consider an info set h of pl i and a feasible action $a \in A(h)$. How should we define $b_i(h)(a)$?
Consider any node x in info set h. The probability of choosing consistent with x is $\pi_i(x)$.
Perfect recall: $\pi_i(x) = \pi_i(y)$ for all $x, y \in h$.
Define

$$b_i(h)(a) = \frac{\pi_i(x,a)}{\pi_i(x)} \quad \text{if } \pi_i(x) > 0 \text{ (and arbitrarily otherwise)}$$

Intuition: conditional on earlier behavior that is consistent with reaching information set h, how likely is i to choose action a?

Example of outcome equivalent strategies



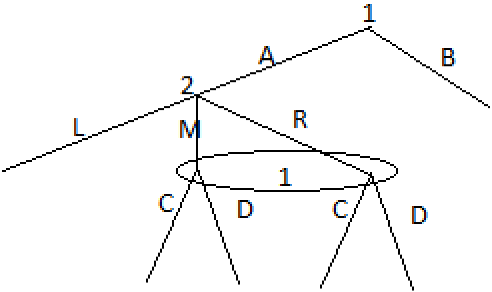
Question: Which behavioral strategies are outcome-equivalent with mixed strategy $(p_{AC}, p_{AD}, p_{BC}, p_{BD})$?

In 1's first information set, the prob that A is chosen is $p_{AC} + p_{AD}$.

In 1's second information set, the prob that C is chosen is computed as the probability of choosing C conditional on earlier behavior that is consistent with this information set being reached:

$$\frac{p_{AC}}{p_{AC} + p_{AD}}. \quad \text{(arbitrary if } p_{AC} + p_{AD} = 0\text{)}$$

Example of outcome equivalent strategies

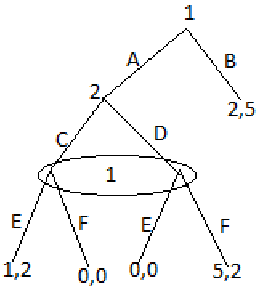


Question: Find a mixed strategy that is outcome equivalent with the behavioral strategy choosing A with prob p and C with prob q.

$$(p_{AC}, p_{AD}, p_{BC}, p_{BD}) = (pq, p(1-q), (1-p)q, (1-p)(1-q))$$

If $p = 0$, the 2nd info set is not reached: end node 'B' is reached with prob 1. Only pure strategies BC and BD are consistent with this node being reached. All mixed strategies with $p_{BC} + p_{BD} = 1$ are then outcome equivalent.

Homework exercise 1



(a) Show that the game above has perfect recall.

(b) For each mixed strategy σ_1 of player 1, find the outcome-equivalent behavioral strategies.

(c) For each behavioral strategy b_1 of player 1, find the outcome-equivalent mixed strategies.

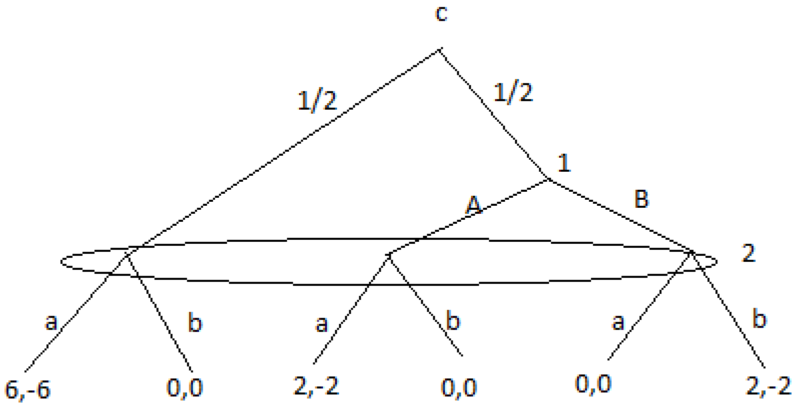
On the definition of strategies

For later, think about the following: pure, mixed, and behavioral strategies specify what happens in *all* information sets of a player. Even in those information sets that cannot possibly be reached if those strategies are used. Why do you think that is the case?

Nash equilibrium

- We can compute, for each profile of pure strategies, the corresponding (expected) payoffs: every extensive form game has a corresponding strategic/normal-form game.
- Terminology: Jörgen used 'normal-form game', the book of Peters uses 'strategic form game'.
- A pure/mixed Nash equilibrium of the extensive form game is then simply a pure/mixed Nash equilibrium of the corresponding strategic form game.
- Nash equilibria in behavioral strategies are defined likewise: a profile of behavioral strategies is a Nash equilibrium if no player can achieve a higher expected payoff by unilaterally deviating to a different behavioral strategy.

Example: from extensive to strategic game



This game (from previous lecture) has strategic form:

	a	b
A	$4, -4$	$0, 0$
B	$3, -3$	$1, -1$

Dominance solvable, unique Nash equilibrium (B, b).

On the definition of strategies

I asked you to think about the following: pure, mixed, and behavioral strategies specify what happens in *all* information sets of a player. Even in those information sets that cannot possibly be reached if those strategies are used. Why do you think that is the case?

Main reason: Nash equilibrium: does each player choose a best reply to the others' strategies?

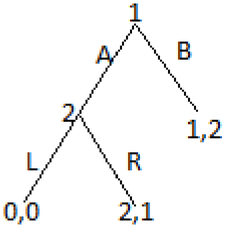
If a player were to deviate, ending up in a different part of the game tree, we need to know what happens there!

Theorem (Equilibrium existence)

Every finite extensive form game with perfect recall has a Nash equilibrium in mixed/behavioral strategies.

1. For mixed strategies: finite extensive form game gives finite strategic game, which has a Nash equilibrium in mixed strategies.
2. For behavioral strategies: by outcome-equivalence, we can construct a Nash equilibrium in behavioral strategies.

The extensive form game



has corresponding strategic form

	L	R
A	$0,0$	$2,1$
B	$1,2$	$1,2$

Pure Nash equilibria: (B, L) and (A, R).

But if pl. 2 is called upon to play, would 2 choose L? This is an implausible choice in the 'subgame' that starts at node A!

To rule out such implausible equilibria, require that an equilibrium is played in each subgame: 'subgame perfect equilibrium'

Subgame perfect equilibrium

- In an extensive form game *with perfect information*, let x be a node of the tree that is not an end node. The part of the game tree consisting of all nodes that can be reached from x is called a *subgame*.
- Each game is a subgame of itself. A subgame on a strictly smaller set of nodes is called a *proper subgame*.
- A *subgame perfect equilibrium* is a strategy profile that induces a Nash equilibrium in each subgame.

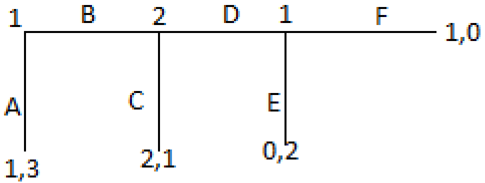
In the game on the previous slide, only (A, R) is subgame perfect.

Subgame perfect equilibria via backward induction

Subgame perfect equilibria are typically found by backward induction:

1. Start with subgames with only one decision left. Determine the optimal actions there.
2. Next, look at subgames with at most two consecutive decisions left. Conditioning on the previous step, the first player to choose (say i) knows what a 'rational' player will do in the subgame that starts after i's choice, so it is easy to find i's optimal action.
3. Continue with subgames of at most 3 consecutive moves, etc.

This is the game-theoretic generalization of the dynamic programming algorithm in optimization theory.



Strategic form:

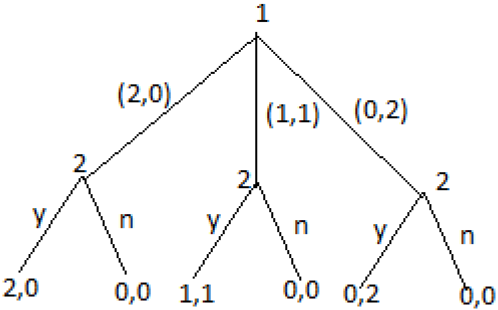
	C	D
AE	$1, 3$	$1, 3$
AF	$1, 3$	$1, 3$
BE	$2, 1$	$0, 2$
BF	$2, 1$	$1, 0$

Pure Nash equilibria: (AE, D), (AF, D), and (BF, C).

Subgame perfect equilibrium: (BF, C)

Backward induction: example 2

Dividing 2 indivisible objects. Pl. 1 proposes, pl. 2 accepts or rejects.



How many pure strategies for player 1? 3
How many pure strategies for player 2? $2^3 = 8$
Subgame perfect equilibria? $((2,0), yyy)$ and $((1,1), nyy)$

Backward induction: example 3 (the 'rotten kid' game)

- A child's action a from some nonempty, finite set A affect both her own payoff $c(a)$ and her parents' payoff $p(a)$; for all $a \in A$ we have $0 \leq c(a) < p(a)$.
- The child is selfish: she cares only about the amount of money she receives.
- Her loving parents care both about how much money they have and how much their child has. Specifically, model the parents as a single player whose utility is the smaller of the amount of money the parents have and the amount the child has. The parents may transfer money to the child (pocket money, trust fund, etc).
- First the child chooses action $a \in A$.
- Then the parents observe the action and decide how much money $x \in [0, p(a)]$ to transfer to the child. The game ends with utility $c(a) + x$ for the child and $\min\{c(a) + x, p(a) - x\}$ to the parents.

Show: in a subgame perfect equilibrium, the child takes an action that maximizes the sum of her private income $c(a)$ and her parents' income $p(a)$. Not so selfish after all!

- In the subgame after action $a \in A$, the parents maximize $\min\{c(a) + x, p(a) - x\}$ over $x \in [0, p(a)]$.
- This is done by choosing x such that $c(a) + x = p(a) - x$, i.e., by $x^*(a) = \frac{1}{2}(p(a) - c(a))$.
- Anticipating this, the child knows that action $a \in A$ leads to transfer $x^*(a)$ and consequently utility $c(a) + x^*(a) = \frac{1}{2}(c(a) + p(a))$. Maximizing this expression is equivalent with maximizing $c(a) + p(a)$.

Finite trees: existence of subgame perfect equilibria

Using backward induction, if there are only finitely many nodes, the first player to move — conditioning on the optimal behavior in the smaller subgames — is optimizing over a finite set: an optimum will always exist. Using this and induction on the 'depth' of the tree, one can show:

Theorem (Existence of subgame perfect equilibria)

In a finite extensive form game with perfect information, there is always a subgame perfect equilibrium in pure strategies.

That's a pretty nice result:

1. no need to consider randomization
2. no implausible behavior in subgames

As an aside: what if there are infinitely many nodes?

Subgame perfection and backward induction coincide

Theorem

In a finite extensive form game with perfect information, subgame perfect equilibria and those found by backward induction are identical.

Difficult! Main step is the 'one-deviation property': a strategy profile is subgame perfect if and only if for each subgame the first player to move cannot obtain a better outcome by changing only the *initial* action.

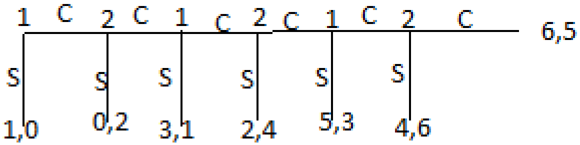
Centipede games

Although subgame perfect equilibria were introduced to rule out implausible behavior in subgames, there are examples where such equilibria lead to outcomes that some people find counterintuitive. This is sometimes corroborated with experimental support. One well-known example consists of Rosenthal's centipede games, characterized by the following properties:

- Players 1 and 2 take turns during at most $2T$ rounds ($T \in \mathbb{N}$).
- At each decision node, the player can choose to (S)top or (C)ontinue.
- The game ends *(i)* if one of the players decides to stop, or *(ii)* if no player has chosen stop after $2T$ periods.
- For each player, the outcome when he stops the game in period t is:
  - *better* than the outcome if the other player stops in period $t+1$ (or the game ends),
  - *worse* than any outcome that is reached if in period $t+1$ the other player continues.

Centipede games

Here is an example of a centipede game with 6 periods:



It is tempting to continue the game if you can be sure that the other player does so as well: the longer the game goes on, the higher the payoffs.

But in the unique subgame perfect equilibrium, players choose (S)top in each node. In particular, the game ends immediately in the initial node.

Reason: in the final node, player 2's best reply is to (S)top. Given that 2 (S)tops in the final round, 1's best reply is to stop one period earlier, etc.

There are other Nash equilibria, but they all lead to the same outcome: player 1 ends the game immediately.

Subgame perfect equilibrium in games with imperfect information

- Subgame perfect equilibria in games with perfect information require each player to play a best reply to other players' strategies in each subgame — regardless of whether that subgame is reached or not.
- It is possible to extend the notion of subgame perfect equilibria to games with imperfect information. But the definition of subgames is trickier: information sets must lie entirely outside the subgame or entirely inside the subgame.
- Formally, let x be a (non-end) node and let V^x be the nodes of the tree that can be reached from x. A well-defined subgame starts at x if and only if each information set h of the original game is a subset of V^x or is a subset of its complement.
- Since extensive form games with imperfect information need not have proper subgames, the notion of subgame perfection typically has little 'bite'.

Homework exercise 2

In the game of homework exercise 1:

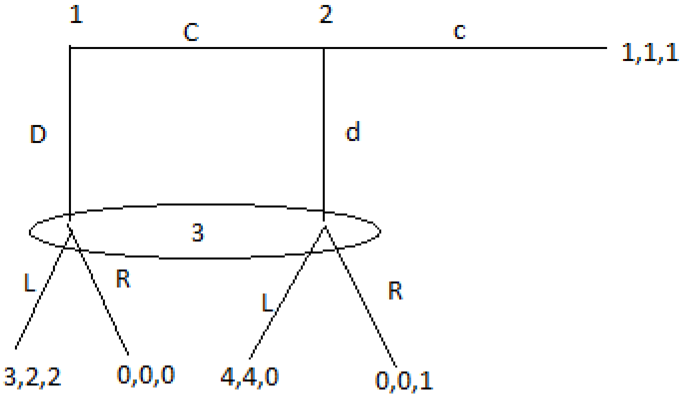
(a) Find the corresponding strategic form game.

(b) Find all pure-strategy Nash equilibria.

(c) What is the outcome of iterated elimination of weakly dominated (pure) strategies?

(d) Find all subgame perfect equilibria (in behavioral strategies).

Implausible behavior in subgame perfect equilibria



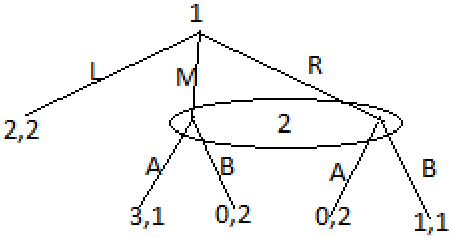
(D, c, L) is a subgame perfect equilibrium. But in 2's information set, c is not a best reply to the others' strategies.
In game-theoretic folklore this game is known as 'Selten's horse'.

Question: Can we find a suitable equilibrium refinement for imperfect information games that

1. makes sense even if there are no subgames and
2. still insists that players choose 'rationally' even in information sets that are reached with zero probability?

First attempt: require best responses in each information set.
Problem: the best response depends on where in the information set the player believes to be!

The following game has no proper subgames:



Beliefs affect optimal strategies: consider pl 2 in info set $\{M, R\}$. A is a best response if and only if the player assigns at most prob $1/2$ to being in node M.

Strategies affect reasonable beliefs: If pl 1 assigns to actions (L, M, R) probabilities $\left(\frac{1}{10}, \frac{3}{10}, \frac{6}{10}\right)$, pl 2 is twice as likely to end up in node R than in node M. Bayes' law requires that he assigns conditional prob $1/3$ to M and $2/3$ to R.

Question: What are reasonable beliefs if 1 chooses L with prob 1?

Assessments

We consider two requirements on beliefs that give different answers to the final question:

1. Weak consistency: in information sets that are reached with positive probability, beliefs are determined by Bayes' law. In information sets reached with zero probability, beliefs are allowed to be arbitrary.
2. Consistency: beliefs are determined as a limit of cases where everything happens with positive probability and — consequently — where Bayes' law can be used.

In particular, in both of these notions, we need to define two things: strategies and beliefs over the nodes in the information sets. The difference will lie in the constraints that are imposed.

Formally, consider a finite extensive form game with perfect recall. An *assessment* is a pair (b, β), where

- $b = (b_i)_{i \in N}$ is a profile of behavioral strategies and
- β is a belief system, assigning to each information set h a probability distribution β_h over its nodes.

Two belief requirements

Given node x and behavioral strategies b, let $\mathbb{P}_b(x)$ be the probability that node x is reached using b: it is the product of the probabilities assigned to the branches leading to x. Similarly, if h is an information set, it is reached with probability $\mathbb{P}_b(h) = \sum_{x \in h} \mathbb{P}_b(x)$.

Assessment (b, β) is:

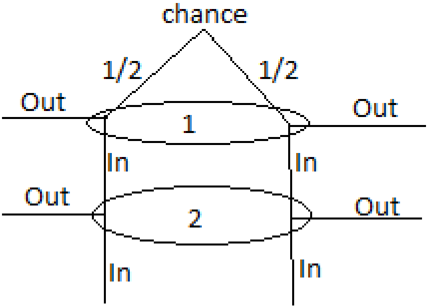
- *weakly consistent* if beliefs in information sets reached with positive probability are determined by Bayes' law:

  $$\beta_h(x) = \mathbb{P}_b(x)/\mathbb{P}_b(h)$$

  for every info set h with $\mathbb{P}_b(h) > 0$ and every node $x \in h$.
- *consistent* if there is a sequence of weakly consistent assessments $(b^m, \beta^m)_{m \in \mathbb{N}}$ with each b^m completely mixed (all actions in all info sets have positive prob) and $\lim_{m \to \infty}(b^m, \beta^m) = (b, \beta)$.

Note: (b, β) consistent $\Rightarrow$ (b, β) weakly consistent.

Consistency: example



In the game above, where payoffs are omitted since they are irrelevant to the question:

(a) Find all weakly consistent assessments (b, β).

(b) Find all consistent assessments (b, β).

Summarize an assessment (b, β) by a 4-tuple $(p, q, \alpha_1, \alpha_2) \in [0,1]^4$, where

- p is the probability that 1 chooses *In*,
- q is the probability that 2 chooses *In*,
- α_1 is the probability that the belief system assigns to the left node in 1's info set,
- α_2 is the probability that the belief system assigns to the left node in 2's info set.

(a) Distinguish two cases:

1. If $p \in (0,1]$, 2's information set is reached with positive probability. In that case, Bayes' Law dictates that $\alpha_1 = \alpha_2 = \frac{1}{2}$. Conclude: all $(p, q, \alpha_1, \alpha_2) \in (0,1] \times [0,1] \times \left\{\frac{1}{2}\right\} \times \left\{\frac{1}{2}\right\}$ are weakly consistent.
2. If $p = 0$, 2's information set is reached with zero probability and 2 is allowed any belief $\alpha_2 \in [0,1]$ over the nodes in the information set. Bayes' Law only dictates that $\alpha_1 = \frac{1}{2}$. Conclude: all $(p, q, \alpha_1, \alpha_2) \in \{0\} \times [0,1] \times \left\{\frac{1}{2}\right\} \times [0,1]$ are weakly consistent.

(b) Every completely mixed profile of behavioral strategies leads to $\alpha_1 = \alpha_2 = \frac{1}{2}$.
Indeed, in 2's information set, both nodes are reached with equal probability $\frac{1}{2}p$.
Conclude: consistent are all
$(p, q, \alpha_1, \alpha_2) \in [0,1] \times [0,1] \times \left\{\frac{1}{2}\right\} \times \left\{\frac{1}{2}\right\}$.

Terminology

In the game theoretic literature there are many versons of consistency requirements on assessments. My version of weak consistency is the same as Definition 27 in Jörgens notes, but is a bit less demanding than Definition 14.12 (Bayesian consistency) in Hans Peters' book. (They coincide whenever all actions are chosen with positive probability, so for defining consistency, either can be used)

Expected payoffs in information sets

Fix assessment (b, β) and an information set h of player i. To formalize the requirement that i plays a best response in info set h, we need to specify i's expected payoff:

1. Conditional on i being in his info set h, belief system β assigns probability $\beta_h(x)$ to being in node $x \in h$.
2. Given such a node x, the probability $\mathbb{P}(e \mid b, x)$ that an end node e is reached, conditional on starting in x and using strategies b is
   - zero if e cannot be reached from x;
   - the product of the probabilities of the corresponding branches from x to e otherwise.
3. In end node e, the payoff to i equals $u_i(e)$.
4. So the expected payoff to agent i in his information set h, given assessment (b, β) is

$$u_i(b_i, b_{-i} \mid h, \beta) = \sum_{x \in h} \beta_h(x) \left(\sum_e \mathbb{P}(e \mid b, x) u_i(e) \right).$$

Sequential rationality

Assessment (b, β) is *sequentially rational* if each player i in each of his information sets h chooses a best response to the belief system β and the strategies of the other players:

$$u_i(b_i, b_{-i} \mid h, \beta) \geq u_i(b_i', b_{-i} \mid h, \beta)$$

for all other behavioral strategies b_i' of player i.

Note:

1. consistency requirements say that beliefs have to make sense given the strategies, without requirements on the strategies;
2. sequential rationality says that strategies have to make sense given the beliefs, without requirements on the beliefs.

Putting the two together, we have:

Weakly perfect Bayesian and sequential equilibrium

An assessment (b, β) is:

- a *weakly perfect Bayesian equilibrium* if it is weakly consistent and sequentially rational,
- a *sequential equilibrium* if it is consistent and sequentially rational.

Sequential equilibria are weakly perfect Bayesian equilibria; the latter are Nash, but not necessarily subgame perfect: they impose no constraints on players' beliefs in subgames off the path of the strategy profile in question.

Theorem (Relations between solution concepts for extensive form games)

(a) *Each finite extensive form game with perfect recall has a sequential equilibrium.*

(b) *If assessment (b, β) is a sequential equilibrium, then b is a subgame perfect equilibrium and (hence) a Nash equilibrium.*

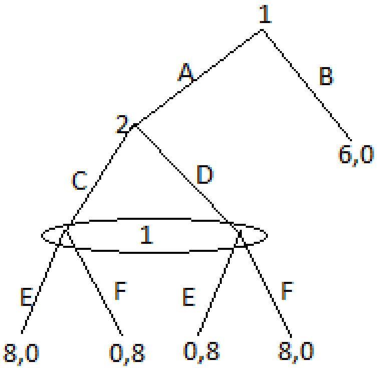
Proof sketch

(a) Via perfect equilibria of an auxiliary 'agent-strategic form game': *extensive-form perfect equilibria!*

- Each player i is split up into agents, one agent for each of i's information sets;
- Agents of i have the same preferences as i;
- A mixed strategy in this agent-strategic form game is a behavioral strategy in the original game;
- Consider a completely mixed seq $b^m \to b$ making b a perfect equilibrium
- For each b^m, Bayes' law gives a belief system β^m.
- Drawing a convergent subsequence if necessary, we can show that $\lim_{m\to\infty}(b^m, \beta^m) = (b, \beta)$ is a sequential equilibrium.

(b) Suppose not. Let i have a profitable deviation b_i' in a subgame starting at some node x. Hence, in this subgame there has to be an information set that is reached with positive probability and where i has a profitable deviation, contradicting sequential rationality and correctness of beliefs.

Compute the sequential equilibria of the game below:



Intuition: What should it be? Player 1 chooses between sure payoffs $(6, 0)$ or the strategic game

	C	D
E	$8, 0$	$0, 8$
F	$0, 8$	$8, 0$

Behavioral strategies $b = (b_1, b_2)$ can be summarized by three probabilities:

1. p, the prob that 1 chooses A in the initial node;
2. q, the prob that 2 chooses C in his information set $\{A\}$;
3. r, the prob that 1 chooses E in information set $\{(A, C), (A, D)\}$.

Belief system β can be summarized by one probability α, the prob assigned to the left node (A, C) in the information set $\{(A, C), (A, D)\}$. Consistency: completely mixed beh. str. have $p, q, r \in (0, 1)$. Bayes' law then gives

$$\alpha = \frac{pq}{pq + p(1-q)} = q,$$

So for each consistent assessment (b, β), it follows that $\alpha = q$. Which of these assessments also satisfies sequential rationality?

Distinguish 3 cases:

1. If $q = 0$, then $\alpha = 0$, so $r = 0$ is 1's unique best reply in the final info set. But if $r = 0$, then $q = 0$ is not a best reply in 2's info set. Contradiction.
2. If $q = 1$, then $\alpha = 1$, so $r = 1$ is 1's unique best reply in the final info set. But if $r = 1$, then $q = 1$ is not a best reply in 2's info set. Contradiction.
3. If $q \in (0, 1)$, rationality in 2's info set $\{A\}$ dictates that both C and D must be optimal. C gives $8(1 - r)$, D gives $8r$, so $r = 1/2$.
   In the info set $\{(A, C), (A, D)\}$ of pl. 1, his expected payoff is
   $$\alpha[8r] + (1 - \alpha)[8(1 - r)] \underbrace{=}_{\alpha = q} 8 - 8q + 8r(2q - 1).$$
   Choosing $r = 1/2$ is rational only if $q = 1/2$.
   Finally, in the initial node, A gives expected payoff 4 and B gives expected payoff 6, so $p = 0$.

Conclude: there is a unique sequential equilibrium with $p = 0, q = r = \alpha = 1/2$.

Homework exercise 3

Find the sequential equilibria (b, β) of the game in homework exercise 1.

- Peters, p. 263: "There is hardly any general method available to compute sequential equilibria: it depends very much on the game at hand what the best way is."
- Some potential approaches are:
  1. Method 1: first find all consistent assessments, then find which of these are sequentially rational.
  2. Method 2: first find all sequentially rational assessments, then find which of these are consistent.
  3. Method 3: by the previous theorem: if (b, β) is a sequential eq, then b is a (subgame perfect) NE. So first find all (subgame perfect) NE. This is easier (no belief system) and often rules out many candidates. Then verify which can be turned into sequential equilibria.
- Methods 1 and 2 are rarely used in that strict order. Often you will discuss sequential rationality and consistency together. If you anyway have to find the subgame perfect equilibria, method 3 can save you a lot of work.

Normal versus extensive form

Given an extensive form game, we defined its Nash equilibria in pure, mixed, behavioral strategies in the corresponding normal-form game. Likewise, it is possible to consider perfect and proper equilibria in this normal-form.

On the other hand, we introduced variants of Nash equilibria (subgame perfect, weakly perfect Bayesian, sequential) that were applicable directly to the extensive form.

We introduced one very important relation: existence of sequential equilibria in the extensive form was derived from the existence of perfect equilibria in the agent-normal form.

There are other links between extensive-form and normal-form analysis, but this is outside the scope of the course; if this interests you, have a look at sections 3.11 till 3.14 of Jörgen's notes and the references there.

Signalling games: examples

1. Michael Spence, 2001 Nobel Memorial Prize in Economics, job-market signalling model
   - A prospective employer can hire an applicant.
   - The applicant has high or low ability, but the employer doesn't know which.
   - Applicant can give a signal about ability, for instance via education.
2. Language, according to some evolutionary biologists, evolved as a way "to tell the other monkeys where the ripe fruit is."

Sometimes it makes sense to signal what your private information is, sometimes not.

Signalling games: model

1. Chance chooses a type t from some nonempty finite set T according to known prob distr $\mathbb{P}$ with $\mathbb{P}(t) > 0$ for all $t \in T$.
2. Pl. 1 (the sender) observes t and chooses a message $m \in M$ in some nonempty finite set of messages M.
3. Pl. 2 (the receiver) observes m (not t) and chooses an action $a \in A$ in some nonempty finite set of actions A.
4. The game ends with utilities $(u_1(t, m, a), u_2(t, m, a))$.

A pure strategy for player 1 is a function $s_1 : T \to M$ and a pure strategy for player 2 is a function $s_2 : M \to A$.

Separating and pooling equilibria in signalling games

In signalling games, it is common to restrict attention to equilibria (s_1, s_2, β), where

- s_1 and s_2 are pure strategies;
- assessment (s_1, s_2, β) is weakly consistent;
- assessment (s_1, s_2, β) is sequentially rational.

Sometimes it is in the sender's interest to try to communicate her type to the receiver by sending different messages for different types

$$s_1(t) \neq s_1(t') \quad \text{for all } t, t' \in T.$$

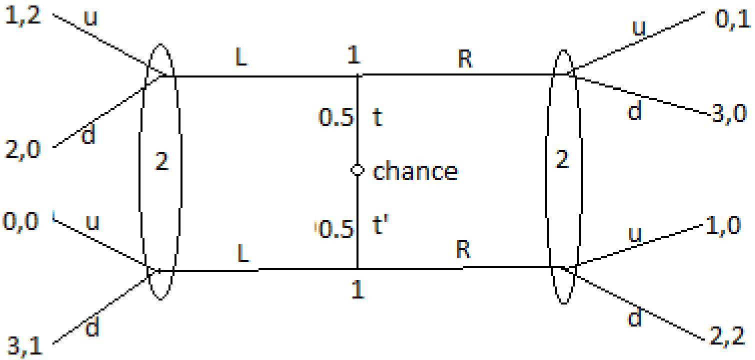
In such cases we call the equilibrium (s_1, s_2, β) a *separating equilibrium*.

In other cases, the sender might want to keep her signal a secret to the receiver and send the same message for each type:

$$s_1(t) = s_1(t') \quad \text{for all } t, t' \in T.$$

In such cases we call the equilibrium (s_1, s_2, β) a *pooling equilibrium*.

Signalling games: example



In the signalling game above:

(a) Find the corresponding strategic form game and its pure-strategy Nash equilibria.

(b) Determine (if any) the game's separating equilibria.

(c) Determine (if any) the game's pooling equilibria.

Answer (a):

- Pl. 1's pure strategies are pairs in $\{L, R\} \times \{L, R\}$, denoting the action after t and t', respectively.
- Pl. 2's pure strategies are pairs in $\{u, d\} \times \{u, d\}$, denoting the action after message L and R, respectively.
- Strategic form:

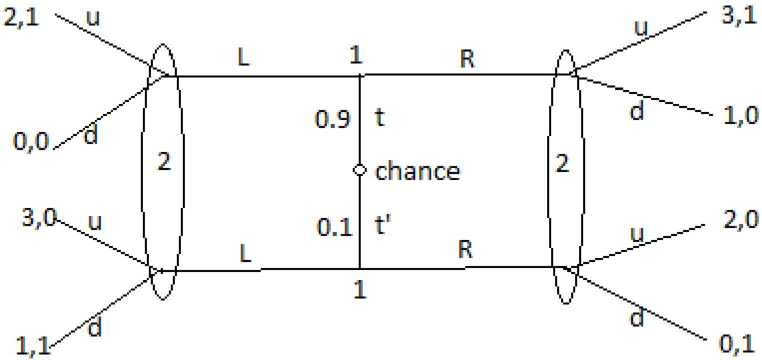
	(u, u)	(u, d)	(d, u)	(d, d)
(L, L)	$\frac{1}{2}, 1^*$	$\frac{1}{2}, 1^*$	$\frac{5}{2}^*, \frac{1}{2}$	$\frac{5}{2}, \frac{1}{2}$
(L, R)	$1^*, 1$	$\frac{3}{2}, 2^*$	$\frac{3}{2}, 0$	$2, 1$
(R, L)	$0, \frac{1}{2}$	$\frac{3}{2}, 0$	$\frac{3}{2}, 1^*$	$3^*, \frac{1}{2}$
(R, R)	$\frac{1}{2}, \frac{1}{2}$	$\frac{5}{2}^*, 1^*$	$\frac{1}{2}, \frac{1}{2}$	$\frac{5}{2}, 1^*$

- Payoffs corresponding with best replies are starred, so there is a unique pure-strategy Nash equilibrium $((R, R), (u, d))$.

Answer (b): Separating equilibria must be Nash equilibria; but the only candidate $((R, R), (u, d))$ is of the pooling type: pl. 1 sends the same message R for both types. Conclude: no separating equilibria.

Answer (c):

- In (a), we found the candidate strategy profile $((R, R), (u, d))$.
- But what should the belief system be? Let $\alpha_1, \alpha_2 \in [0, 1]$ denote the prob assigned to the top node in the left and right info set, respectively.
- Bayesian consistency: requires that $\alpha_2 = \frac{1}{2}$, but imposes no constraints on α_1.
- Sequential rationality:
  1. Both info sets of pl. 1 and the right info set of pl. 2 are reached with positive prob. Since $((R, R), (u, d))$ is a NE, the players choose a best reply in those information sets.
  2. The left info set of pl. 2 is reached with zero prob. But the beliefs should be such that 2's action u is a best reply there.
  3. Pl. 2's payoff from u is $2\alpha_1 + 0(1 - \alpha_1)$ and from d is $0\alpha_1 + 1(1 - \alpha_1)$, so seq. rat. requires $\alpha_1 \geq \frac{1}{3}$.
- Conclude: Assessments (s_1, s_2, β) with strategies $(s_1, s_2) = ((R, R), (u, d))$ and belief system $\beta = (\alpha_1, \alpha_2) \in [1/3, 1] \times \{1/2\}$ are the game's pooling equilibria.



In the signalling game above:

(a) Find the corresponding strategic form game and its pure-strategy Nash equilibria.

(b) Determine (if any) the game's pooling equilibria.

(c) Determine (if any) the game's separating equilibria.

Reading guide

- definition extensive form games: slides 1–5, book §4.1, §14.1
- examples (im)perfect recall: slides 6–8, book 53–54, 253
- pure, mixed, behavioral strategies: slides 9–12, book §4.2, §14.2
- outcome equivalence of mixed and behavioral strategies under perfect recall: slides 13–19, book §14.2
- Nash equilibrium: slides 21–24, book §4.2, 256–258
- subgame perfect equilibrium and backward induction: slides 25–38, book §4.3, §14.3.1
- assessments: consistency of beliefs: slides 39–46, book §4.4 (partly), §14.3.2 (partly), Jörgen's notes §3.9
- assessments: sequential rationality: slides 47–48, book p. 259
- weakly perfect Bayesian and sequential equilibrium: slides 49–55, book §4.4, §14.3.2, Jörgen's notes §3.9
- signalling games: slides 57–63, book §5.3