

SF2972 GAME THEORY

Normal-form analysis I

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1 Normal form games

Definition 1.1 A normal-form game is a triplet $G = \langle I, S, f \rangle$, where

(i) I is the set of **players**

(ii) $S = \times_{i \in I} S_i$ is the set of **strategy profiles** $s = (s_i)_{i \in I}$ with S_i denoting the strategy set of each player $i \in I$

(iii) $f : S \rightarrow \mathbb{R}^{|I|}$ is the **combined payoff function**, where $f_i(s) \in \mathbb{R}$ is the payoff to player i when strategy profile s is played

- Notation: for any strategy profile $s \in S$, player $i \in I$ and strategy $s'_i \in S_i$, $f_i(s'_i, s_{-i}) \in \mathbb{R}$ denotes the payoff that player i obtains when playing s'_i while all others stick to their strategies in s

Definition 1.2 A finite normal-form game is a normal-form game $G = \langle I, S, f \rangle$ in which S is finite. Then $I = \{1, 2, \dots, n\}$ is the finite set of players and $S_i = \{1, 2, \dots, m_i\}$ is player's i 's finite set of strategies.

- Game theory treats payoff functions as primitives
- Each player is assumed to strive to maximize the expected value of his or her payoff. In economics the payoffs are derived from the economic agents' Bernoulli functions
- It is usually presumed that each player can randomize over his or her strategy set. Different players' randomizations are then assumed to be statistically independent. We will call this *the mixed-strategy extension* of the game

Example 1.1 (Firm-worker game) Consider a firm owner, player 1, who offers a wage $w \in W = \{1, 2, \dots, 99, 100\}$ to a worker, player 2. The worker has a binary choice, to either accept the offer, $a = 1$, or reject it, $a = 0$. In the first case, the owner makes a profit of $100 - w$ and the worker earns income w , while in the second case the owner makes zero profit and the worker earns income $w_0 = 30$ (her outside option or reservation wage). Define this as a normal-form game! What if $W = [1, 100]$?

2 Mathematics reminder

- **Notation:** $\mathbb{N}$ the positive integers, $\mathbb{R}$ the reals, $\mathbb{R}_+$ the non-negative reals
- **Standard definitions we will use:** *open, closed, convex and bounded* sets $X \subseteq \mathbb{R}^n$, the *interior, closure* and *convex hull* of arbitrary sets $X \subseteq \mathbb{R}^n$
 - *upper-contour sets* $\{x \in X : f(x) \geq \alpha\}$ for functions $f : X \rightarrow \mathbb{R}$
 - *quasi-concave functions* $f : X \rightarrow \mathbb{R}$, with X convex: all its upper contour-sets are convex
- **Notation:** Given any function $f : X \rightarrow \mathbb{R}$, its set of *maximands*:

$$X^* = \arg \max_{x \in X} f(x) = \{x^* \in X : f(x^*) \geq f(x) \ \forall x \in X\}$$

Theorem 2.1 (Weierstrass' Maximum Theorem) *If $X \subset \mathbb{R}^n$ is non-empty and compact and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous, then $X^* = \arg \max_{x \in X} f(x)$ is non-empty and compact.*

Definition 2.1 *A fixed point under a function $f : X \rightarrow X$ is a point $x^* \in X$ such that $x^* = f(x^*)$.*

Theorem 2.2 (Brouwer's Fixed Point Theorem) *If $X \subset \mathbb{R}^n$ is non-empty, compact and convex and $f : X \rightarrow X$ is continuous, then f has at least one fixed point.*

Definition 2.2 A correspondence φ from a set X to a set Y , written $\varphi : X \rightrightarrows Y$, is a "rule" that assigns to each element $x \in X$ a unique non-empty subset $\varphi(x) \subseteq Y$.

Definition 2.3 A correspondence $\varphi : X \rightrightarrows Y$, for $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ open, is **upper hemi-continuous (u.h.c.)** at $x \in X$ if for every open set B such that $\varphi(x) \subset B$ there exists an open set A containing x such that $\varphi(x') \subset B \forall x' \in A \cap X$.

- A correspondence is upper hemi-continuous if it is u.h.c. at all points in its domain X .

- Economic agents in general, and players in games in particular, are assumed to choose their actions/plans/strategies so as to maximize their expected utility, profit or payoff. More generally, consider

$$\max_{x \in X} f(x, y)$$

where $f : \mathbb{R}^{n+m} \rightarrow \mathbb{R}$, $X \subseteq \mathbb{R}^n$ (say own action, plan, or strategy) and $y \in \mathbb{R}^m$ (say, others' actions, plans, or strategies) for some $n, m \in \mathbb{N}$.
Let

$$\xi(y) = \arg \max_{x \in X} f(x, y) = \{x^* \in X : f(x^*, y) \geq f(x, y) \quad \forall x \in X\} \quad (1)$$

- A special case of Berge's Maximum Theorem:

Proposition 2.3 (Berge) *If f is continuous and X is nonempty and compact, then (1) defines an u.h.c. and compact-valued correspondence $\xi : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$.*

Definition 2.4 A fixed point under a correspondence $\varphi : X \rightrightarrows X$ is a point $x^* \in X$ such that $x^* \in \varphi(x^*)$.

Theorem 2.4 (Kakutani's Fixed-Point Theorem) If $X \subset \mathbb{R}^n$ is non-empty, compact and convex, and $\varphi : X \rightrightarrows X$ is convex-valued, compact-valued and upper hemi-continuous, then φ has at least one fixed point.

3 Finite normal-form games

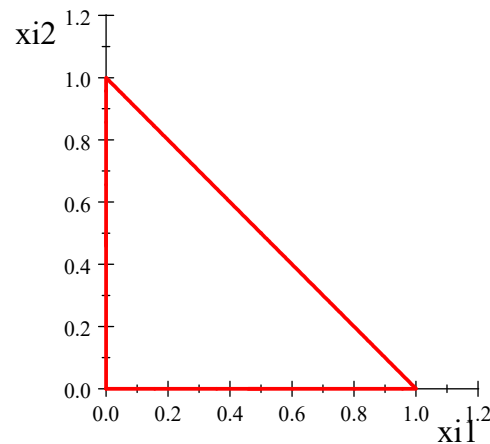
- Recall that a **finite normal-form game** is a normal-form game $G = \langle I, S, u \rangle$ in which S is finite. Then $I = \{1, 2, \dots, n\}$ is the finite set of *players* and $S_i = \{1, 2, \dots, m_i\}$ is player's i 's finite set of pure *strategies*.

3.1 The mixed-strategy extension of a finite game

- For each player i in a finite game $G = \langle I, S, u \rangle$, the mixed-strategy set is

$$\Delta_i = \Delta(S_i) = \{x_i \in \mathbb{R}_+^{m_i} : \sum_{h=1}^{m_i} x_{ih} = 1\},$$

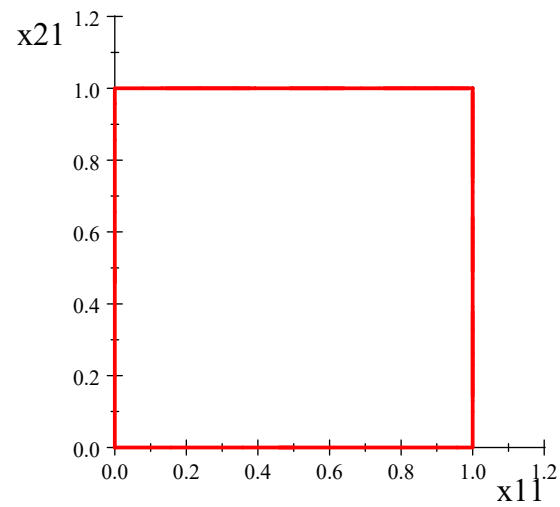
the unit simplex "spanned" by the player's pure strategies



- The *vertices* of Δ_i are the *unit vectors*, $\mathbf{1}_{ih} = (0, 0, \dots, 0, 1, 0, \dots, 0, 0)$ with the 1 in position h
- The (relative) interior: $\text{int}(\Delta_i) = \{x_i \in \Delta_i : x_{ih} > 0 \ \forall h \in S_i\}$, these are mixed strategies with *full support*, called *interior* or *completely mixed* strategies

- The *polyhedron* of mixed-strategy *profiles*:

$$\square(S) = \times_{i \in I} \Delta_i = \times_{i \in I} \Delta(S_i)$$



- For each mixed-strategy profile $x \in \square(S)$ and player $i \in I$, let $\tilde{u}_i(x) \in \mathbb{R}$ be the *expected value* of $u_i(s)$ when players use their mixed strategies in x :

$$\tilde{u}_i(x) = \sum_{s \in S} \left[\left(\prod_{i=1}^n x_i(s_i) \right) \cdot u_i(s) \right]$$

- This defines the mixed-strategy extension $\tilde{G} = \langle I, \square(S), \tilde{u} \rangle$ of the finite game $G = \langle I, S, u \rangle$ as a normal-form game with convex and compact strategy sets, Δ_i , and continuous (in fact multi-linear) payoff functions $\tilde{u}_i$
- In particular, each payoff $\tilde{u}_i(x)$ is linear in the player's own mixed strategy, x_i , for any given profile x_{-i} for the other players. To see this, note that for all $x \in \square(S)$:

$$\tilde{u}_i(x) = \sum_{h \in S_i} \tilde{u}_i(\mathbf{1}_{ih}, x_{-i}) \cdot x_{ih}$$

4 Dominance relations and best replies

4.1 Dominance relations

Definition 4.1 $x_i^* \in \Delta_i$ strictly dominates $x_i' \in \Delta_i$ if

$\tilde{u}_i(x_i^*, x_{-i}) > \tilde{u}_i(x_i', x_{-i})$ for all $x \in \square(S)$.

Definition 4.2 $x_i^* \in \Delta_i$ weakly dominates $x_i' \in \Delta_i$ if $\tilde{u}_i(x_i^*, x_{-i}) \geq \tilde{u}_i(x_i', x_{-i})$ for all $x \in \square(S)$ with $>$ for some $x \in \square(S)$.

Definition 4.3 $x_i^* \in \Delta_i$ is weakly dominant if it weakly dominates all strategies $x_i' \in \Delta_i$.

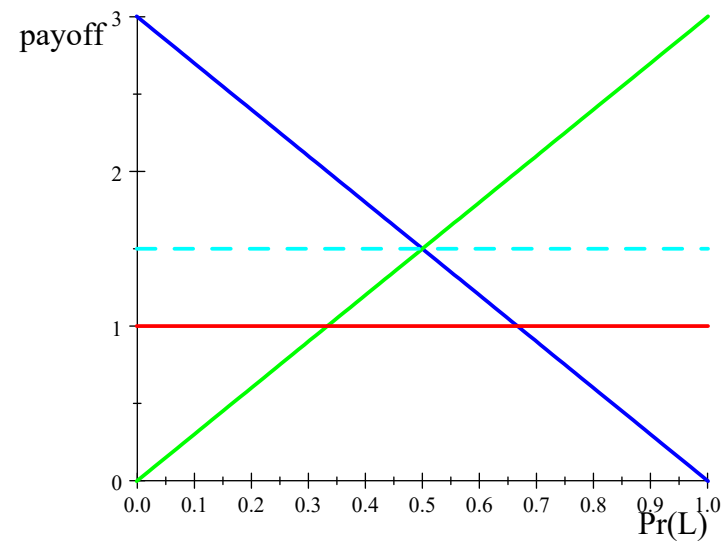
- A strategy that is not weakly dominated is called *undominated*

Example 4.1 *Consider the game*

	a'	b'
a	1, 1	2, 0
b	0, 0	0, 0

Example 4.2 Consider the game

3, a	0, b
0, c	3, d
1, e	1, f



Remark:

- The conclusion that a rational player will never use pure strategy B may seem “too strong” since strategies T and M “look risky”, while B is safe
- However, the payoff numbers in games are the players’ Bernoulli function values (see the von Neumann - Morgenstern’s expected utility hypothesis and its axioms), so the claim in fact makes good sense

- Suppose, for instance that the above game is an interaction with monetary gains and that player 1 is selfish but perhaps risk averse:
 - pure strategy profiles (T, L) and (M, R) each result in 1000 euros for player 1
 - pure strategy profiles (T, R) and (M, L) result in 0 euros for player 1
 - pure strategy profiles (B, L) and (B, R) both result in z euros to player 1
- We can now determine z from the given payoff numbers! (In class). Set Bernoulli function values $v(0) = 0$, $v(1000) = 3$ and $v(z) = 1$. Player 1 should be indifferent between getting z for sure and a lottery with probability $1/3$ for 1000 euros and probability $2/3$ for zero euros. Find z for yourself as player 1 and now ask yourself if you prefer getting z for sure or the lottery $\frac{1}{2}[T] + \frac{1}{2}[M]$

- *Iterated elimination* of strictly dominated pure strategies:

$$G = \begin{bmatrix} 3, 3 & 1, 0 & 6, 1 \\ 0, 1 & 0, 0 & 4, 2 \\ 1, 6 & 2, 4 & 5, 5 \end{bmatrix}$$

- One can show, in general, that the order of elimination of *strictly* dominated strategies does not matter for the end results. The resulting non-empty subset of pure strategies, $\emptyset \neq Q_i \subseteq S_i$, one for each player i , is always the same.

Definition 4.4 *A game is dominance solvable if Q is a singleton set.*

- The epistemic foundations for avoidance of *weakly* dominated strategies are fragile [see Samuelson (1992), Brandenburger, Friedenberg and Keisler (2008), Hillas and Samet (2014)]

Q: If I know that others avoid weakly dominated strategies, then why should I do it?

Example 4.3

	a'	b'
a	1, 1	1, 0
b	1, 0	0, 0

4.2 Best replies

- The i :th player's *pure-strategy best-reply correspondence* $\beta_i : \square(S) \rightrightarrows S_i$ is defined by

$$\beta_i(x) = \{h \in S_i : \tilde{u}_i(\mathbf{1}_{ih}, x_{-i}) \geq \tilde{u}_i(\mathbf{1}_{ik}, x_{-i}) \ \forall k \in S_i\}$$

- Mixed strategies cannot give higher payoffs than pure: let

$$\gamma_i(x) = \{h \in S_i : \tilde{u}_i(\mathbf{1}_{ih}, x_{-i}) \geq \tilde{u}_i(x'_i, x_{-i}) \ \forall x'_i \in \Delta_i\}$$

then $\gamma_i(x) = \beta_i(x)$

- The i :th player's *mixed-strategy best-reply correspondence* $\tilde{\beta}_i : \square(S) \rightrightarrows \Delta_i$ is defined by:

$$\tilde{\beta}_i(x) = \{x_i^* \in \Delta_i : \tilde{u}_i(x_i^*, x_{-i}) \geq \tilde{u}_i(x'_i, x_{-i}) \ \forall x'_i \in \Delta_i\}$$

- Note that $\tilde{\beta}_i(x)$ is a (non-empty) subsimplex
- The **combined pure *BR*-correspondence** $\beta : \square(S) \rightrightarrows S$:

$$\beta(x) = \times_{i \in I} \beta_i(x)$$

- The **combined mixed *BR*-correspondence** $\tilde{\beta} : \square(S) \rightrightarrows \square(S)$:

$$\tilde{\beta}(x) = \times_{i \in I} \tilde{\beta}_i(x)$$

4.3 Dominance vs. best replies

- Pure best replies are not strictly dominated
- If a pure strategy is not strictly dominated is it then a best reply to some belief?
- Pure best replies to *interior* strategy profiles are undominated
- If a pure strategy is undominated is it then a best reply to some interior belief?

Proposition 4.1 (Pearce, 1984) Suppose $n = 2$. Then $s_i \in S_i$ is not strictly dominated iff $s_i \in \beta_i(x)$ for some $x \in \square(S)$, and $s_i \in S_i$ is undominated iff $s_i \in \beta_i(x)$ for some $x \in \text{int}(\square(S))$.

5 Rationalizability

- Consider a finite game in normal form, $G = \langle I, S, u \rangle$ and assume

A1 (*Rationality*): Each player i forms a probabilistic belief $\mu_j^i \in \Delta(S_j)$ about every other player j 's strategy choice, a belief that does not contradict any information or knowledge that player i has, and player i chooses a (pure or mixed) strategy that maximize his or her expected payoff, assuming statistical independence between other player's strategy choices.

A2 (*Common Knowledge*): The game G and the players' rationality (A1) is common knowledge among the players: each player knows G and that (A1) holds for all players, knows that all players know this, and knows that all players know that all players know this etc. *ad infinitum*.

- Question: What is the logical implication of A1 and A2?

- Answer: rationalizability!

1. For any $X_j \subseteq \Delta(S_j)$, let $X = \times_{j=1}^n X_j$ and write

$$\tilde{\beta}_i(X) = \left\{ x_i^* \in \Delta(S_i) : x_i^* \in \tilde{\beta}_i(x) \text{ for some } x \in X \right\}$$

2. Write $X_j(0) = \Delta(S_j)$ and $X(0) = \times_{j=1}^n X_j(0)$. Thus $X(0) = \square(S)$

3. Define the set sequence $\langle X(t) \rangle_{t \in \mathbb{N}}$ recursively by

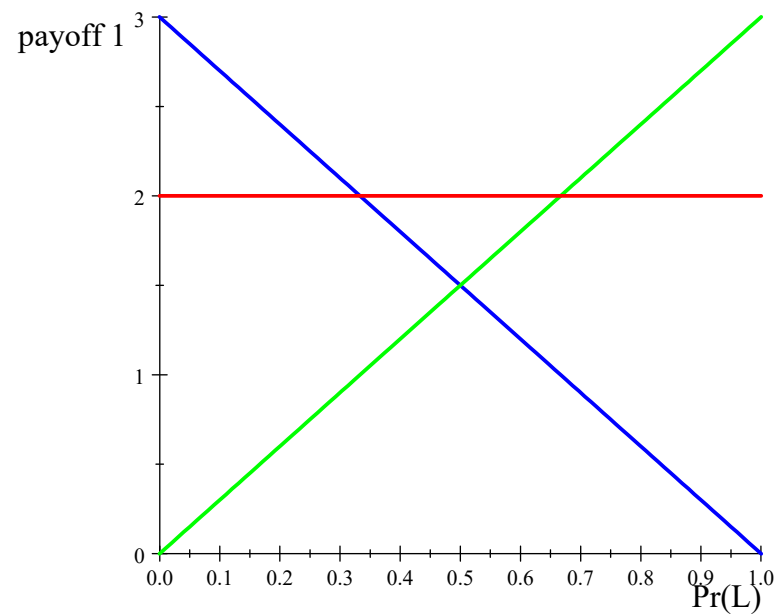
$$X_i(t+1) = \tilde{\beta}_i[C(t)]$$

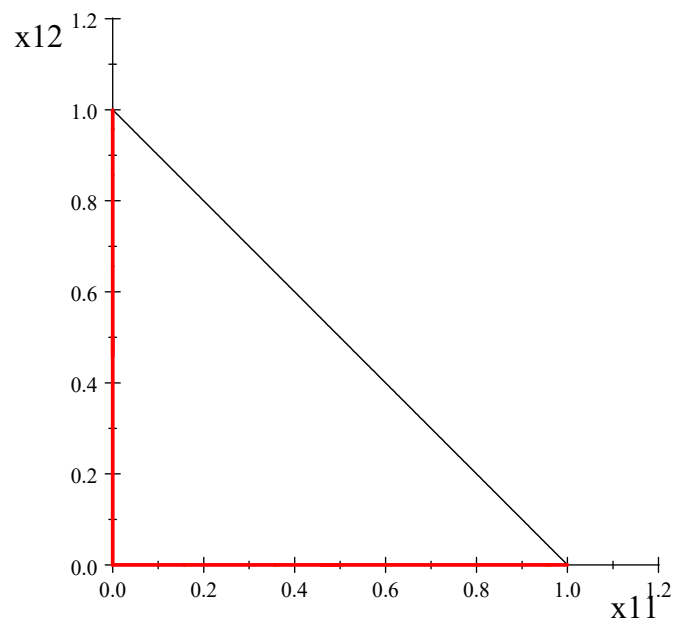
where $X(t) = \times_{j=1}^n X_j(t)$, $C(t) = \times_{j=1}^n C_j(t)$ and

$C_j(t)$ is the *convex hull* of $X_j(t)$

- Note that $X_i(t+1) \subseteq X_i(t)$ for all t and i
- Note that a set $X_i(t)$ is not necessarily convex:

$$\begin{array}{ll} 3, a & 0, b \\ 0, c & 3, d \\ 2, e & 2, f \end{array}$$





$$X_1 = \{x_1 \in \Delta_1 : x_{11}x_{12} = 0\} \quad \neq \quad C_1 = \Delta_1$$

Definition 5.1 (Pearce, 1984) A strategy $x_i \in \Delta(S_i)$ is rationalizable for player i if $x_i \in X_i^\infty$, where

$$X_i^\infty = \cap_{t \in \mathbb{N}} X_i(t).$$

- Let C_i^∞ be the convex hull of X_i^∞

Proposition 5.1 For each i : $X_i^\infty \neq \emptyset$ and $C_i^\infty = \Delta(R_i)$ for a non-empty subset $R_i \subseteq S_i$, the rationalizable pure strategies for player i .

- Let $R = \times_{i \in I} R_i$ be the set of rationalizable pure-strategy profiles, and let $Q = \times_{i \in I} Q_i$ be the set of pure-strategy profiles that survive the iterated elimination of strictly dominated pure strategies. Then

$$R \subseteq Q$$

for any finite game, with equality if $n = 2$. (For $n > 2$, it is in some games more demanding to be rationalizable, since one has to find an independent probability distribution across two or more other players.)

6 Nash equilibrium

Definition 6.1 A mixed-strategy profile $x \in \Delta(S)$ is a **Nash equilibrium (NE)** if $x \in \tilde{\beta}(x)$

- A NE x is *strict* if $\tilde{\beta}(x) = \{x\}$
- A NE strategy cannot be *strictly* dominated, but may be weakly dominated

Definition 6.2 A Nash equilibrium x is **undominated** if no strategy component x_i is weakly dominated.

6.1 Existence

Theorem 6.1 (Nash, 1950) *The mixed-strategy extension $\tilde{G}$ of any finite game has at least one Nash equilibrium.*

Two alternative proofs:

1. Application of Kakutani's fixed-point theorem (Nash's first proof)
2. Application of Brouwer's fixed-point theorem (Nash's second proof).
This inspired Arrow's and Debreu's proof of the existence of Walrasian equilibrium in general-equilibrium theory.

- **Proof 1 (1950):** The polyhedron $\square(S)$ is non-empty, convex and compact. Berge's Maximum Theorem implies that $\tilde{\beta} : \square(S) \rightrightarrows \square(S)$ is upper hemi-continuous. We saw that $\tilde{\beta}(x)$ is a non-empty convex and closed set. Hence, Kakutani's Fixed-Point Theorem applies, so $x^* \in \tilde{\beta}(x^*)$ for at least one $x^* \in \square(S)$.
- **Proof 2 (1951):** Let $u_{ih}^+(x) = \max\{0, \tilde{u}_i(1_{ih}, x_{-i}) - \tilde{u}_i(x)\}$, the "excess payoff" to pure strategy h , and define $f : \square(S) \rightarrow \square(S)$ by

$$f_{ih}(x) = \frac{x_{ih} + u_{ih}^+(x)}{1 + \sum_{k \in S_i} u_{ik}^+(x)} \quad \forall i \in I, h \in S_i$$

Clearly f is continuous and thus has a fixed point by Brouwer's Fixed-Point Theorem. Not difficult to verify that each fixed point x^* is a NE.