SF2972 GAME THEORY
Normal-form analysis II

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1 Nash equilibrium

Domain of analysis: finite NF games $G = \langle I, S, u \rangle$ with mixed-strategy extension $\tilde{G} = \langle I, \boxdot(S), \tilde{u} \rangle$

Definition 1.1 *A strategy profile $x \in \boxdot(S)$ is a* **Nash Equilibrium (NE)** *if $x \in \tilde{\beta}(x)$.*

- Note that a strategy profile x is a NE if and only if

$$x_{ih} > 0 \quad \Rightarrow \quad h \in \beta_i(x) \quad \forall i \in I, h \in S_i$$

- and note also that this is equivalent with the condition that $h \notin \beta_i(x) \quad \Rightarrow \quad x_{ih} = 0$.

1.1 Invariance properties

1. Positive affine transformations of any player's payoffs: $v_i = \alpha_i u_i + \lambda_i$ for any $\alpha_i > 0$ and $\lambda_i \in \mathbb{R}$

2. "Local shifts" of a player's payoffs: add any constant $\theta_{ijk} \in \mathbb{R}$ to all i's payoff whenever some other player j plays some strategy $k \in S_j$

3. Elimination of strictly dominated strategies

4. Elimination of non-rationalizable strategies

- When solving a game for NE, always first try to simplify the game by way of these transformations!

Example 1.1 *Simplify and solve for NE!*

$$\begin{array}{ccc} & L & R \\ T & 5,5 & 0,4 \\ B & 4,0 & 2,2 \\ D & 4,7 & 1,8 \end{array}$$

1.2 Implausible Nash equilibria

- The entry-deterrence game: infinitely many arguably implausible equilibria

- The firm-worker game: infinitely many arguably implausible equilibria

- What about the following game?

$$\begin{array}{ccc} & L & R \\ T & 9,9 & 0,0 \\ B & 0,0 & 0,0 \end{array}$$

- Can one discard implausible Nash equilibria by first principles?

- We will study two refinements: *perfection* (Selten, 1975) and *properness* (Myerson, 1978)

2 Perfect equilibrium

The probably most well-known refinement of Nash equilibrium is that of *"trembling hand" perfection*, due to Selten (1975).

- Selten (1975): "Rationality as the limit of bounded rationality when the bounds are gradually lifted"

- Players have "trembling hands," and know this!

- Imagine that players sometimes, maybe very rarely, make mistakes and are aware of this risk, for themselves and others

- Recall that a strategy profile x is a NE iff $h \notin \beta_i(x) \Rightarrow x_{ih} = 0$, that is, suboptimal pure strategies are not used at all

- Recall that a strategy profile x is interior if $x_{ih} > 0 \; \forall i \in I, h \in S_i$

- The following definition is equivalent to Selten's original definition:

Definition 2.1 *Given $\varepsilon > 0$, an interior strategy profile $x \in int\left[\boxdot\left(S\right)\right]$ is* ε-**perfect** *if $x \in int\left[\boxdot\left(S\right)\right]$ and*

$$h \notin \beta_i\left(x\right) \quad \Rightarrow \quad x_{ih} \leq \varepsilon$$

A **perfect equilibrium** *is any limit of ε-perfect strategy profiles as $\varepsilon \to 0$.*

1. Claim: PE $\Rightarrow$ NE. [Let x^* be a PE and suppose $h \notin \beta_i\left(x^*\right)$. Then $\tilde{u}_i(\mathbf{1}_{ih}, x^*_{-i}) < \tilde{u}_i(x^*)$ so by continuity $\tilde{u}_i(\mathbf{1}_{ih}, x_{-i}) < \tilde{u}_i(x) \; \forall x$ sufficiently close to x^*. Hence, $h \notin \beta_i\left(x\right)$, and thus $x^{\varepsilon}_{ih} \to 0$.]

2. Claim: All completely mixed Nash equilibria are perfect. [Every such strategy profile x^* is ε-perfect for any $\varepsilon > 0$]

Theorem 2.1 (Selten, 1975) *The mixed-strategy extension $\tilde{G}$ of any finite normal-form game G has at least one perfect equilibrium.*

- This existence result will be a corollary to a later result.

- Characterization of perfection in terms of robustness to strategic uncertainty:

Proposition 2.2 (Selten, 1975) *x^* is a perfect equilibrium $\Leftrightarrow$ every neighborhood of x^* contains some $x \in int(\boxdot)$ such that $x^* \in \tilde{\beta}(x)$.*

- $\Rightarrow$ Every strict Nash equilibrium is perfect (then each player's strategy is, by continuity, the unique best reply to all nearby interior profiles)

- Moreover:

Corollary 2.3 *If x^* is a perfect equilibrium, then x^* is undominated.*

Proof: Suppose that $x_i^* \in \Delta_i$ is weakly dominated by some strategy $\tilde{x}_i \in \Delta_i$. Then x_i^* is not a best reply to any $x \in int(\boxdot)$. **Q.E.D.**

- In fact, *all* undominated Nash equilibria in two-player games are perfect:

Proposition 2.4 (van Damme, 1987) *If x is an undominated Nash equilibrium in a two-player game, then x is a perfect equilibrium.*

- Counter-example when $n = 3$ and each player has 2 pure strategies. Let player 1 choose row, player 2 column, and player 3 trimatrix (M or K):

$$\begin{array}{ccc} & L & R \\ T & 1,1,1 & 1,0,1 \\ B & \mathbf{1,1,1} & 0,0,1 \\ & M & \end{array} \qquad \begin{array}{ccc} & L & R \\ T & 1,1,0 & 0,0,0 \\ B & 0,1,0 & 1,0,0 \\ & K & \end{array}$$

  - $s = (B, L, M)$ is clearly an undominated NE. But it is not perfect ($s_1 = B$ non-robust against 3's trembles.) The unique PE is $s^* = (T, L, M)$.

Example 2.1 *The entry-deterrence game*

	C	F
A	$1, 3$	$1, 3$
E	$2, 2$	$0, 0$

Perfection rules out all implausible Nash equilibria!

Example 2.2 *Reconsider the firm-worker example. Thus,* $G = \langle \{1, 2\}, W \times F, u \rangle$*, where* $W = \{1, 2, ..., 100\}$ *and* F *is the set of functions from* W *to* $\{0, 1\}$*. We noted before that* $W^{NE} = W \cap [30, 100]$*. Yet only* $w = 30$*, and perhaps also* $w = 31$*, "make sense" as predictions for wages that may be agrees upon. And indeed:* $W^{PE} = W \cap \{30, 31\}$*. Again perfection rules out all implausible NE!*

Example 2.3 *Reconsider the game*

$$\begin{array}{ccc} & L & R \\ T & 7,7 & 0,0 \\ B & 0,0 & 0,0 \end{array}$$

However...

- Myerson (1978) pointed out that perfection is sensitive to the addition of a strictly dominated strategy —an arguably undesirable property of a solution concept.

Example 2.4 *Add a "dumb" strategy to the entry-deterrence game (say, the potential entrant may shoot himself in the foot):*

	C	F
A	$1, 3$	$\mathbf{1, 3}$
E	$2, 2$	$0, 0$
D	$-4, -1$	$-4, 0$

Before, only $s^* = (E, C)$ was perfect. But now also $s^o = (A, F)$ becomes perfect! Because F is no longer weakly dominated.

3 Proper equilibrium

- Myerson (1978): People are less likely to make more costly mistakes, so we should require some "order" among mistake probabilities:

Definition 3.1 (Myerson, 1978) *Given $\varepsilon > 0$, an interior strategy profile $x \in int\left[\boxdot\left(S\right)\right]$ is ε-**proper** if $x \in int\left[\boxdot\left(S\right)\right]$ and*

$$\tilde{u}_i\left(\mathbf{1}_{ih}, x_{-i}\right) < \tilde{u}_i\left(\mathbf{1}_{ik}, x_{-i}\right) \quad \Rightarrow \quad x_{ih} \leq \varepsilon \cdot x_{ik}$$

A **proper equilibrium** *is any limit of ε-proper strategy profiles as $\varepsilon \to 0$.*

- Every ε-proper strategy profile is ε-perfect, so every proper equilibrium is perfect!

- Every completely mixed NE is ε-proper for all $\varepsilon > 0$. Hence all such equilibria are proper

- But is it to ask for too much to ask for properness?

Proposition 3.1 (Myerson, 1978) *The mixed-strategy extension $\tilde{G}$ of any finite normal-form game G has at least one proper equilibrium.*

- This result follows from the Bolzano-Weierstrass Theorem if for every $\varepsilon > 0$ sufficiently small there exists an ε-proper strategy profile. Hence, it remains to establish existence of ε-proper strategy profiles for arbitrary small $\varepsilon > 0$.

- Once the existence of proper equilibria has been established, the existence of perfect (and in fact also Nash) equilibria follows

Proof sketch for Proposition 3.1:

1. Let $\varepsilon \in (0,1)$

2. Ask each player i to submit a strict and complete *ranking* of his or her m_i pure strategies

3. For each player i, a computer will pick i's pure strategy with rank r with probability

$$p_r = \frac{\varepsilon^r}{\varepsilon + \varepsilon^2 + \varepsilon^3 + .. + \varepsilon^{m_i}} \text{ for } r = 1, 2, ..., m_i$$

4. This defines a finite metagame G^* in which a pure strategy is a ranking (of one's pure strategies in G)

5. G^* being finite, its mixed-strategy extension has at least one NE. Any such metagame strategy-profile induces an ε-proper strategy profile in $\tilde{G}$

Example 3.1 *The augmented entry-deterrence game*

$$\begin{array}{ccc} & C & F \\ A & 1,3 & \mathbf{1,3} \\ E & 2,2 & 0,0 \\ D & -4,-1 & -4,0 \end{array}$$

While $s^o = (A, F)$ is perfect, it is not proper because D is a more costly mistake for player 1 than E when play is close to (A, F).

- Properness has an amazing implication for extensive-form analysis - a topic we will take up after we have defined perfect and sequential equilibria in extensive-form games

4 Payoff-equivalent strategies and the reduced normal form

- A normal-form game may contain two or more pure strategies that result in exactly the same payoffs to *all* players.

Definition 4.1 *Two pure strategies* $s_i', s_i'' \in S_i$ *in a normal-form game are* payoff equivalent *if* $u\left(s_i', s_{-i}\right) = u\left(s_i'', s_{-i}\right)$ *for all pure-strategy profiles* $s \in S$.

- Note that the whole *payoff vector* (with one component for every player) has to remain unchanged if player i were to switch from strategy s_i' to strategy s_i''

- For each player $i \in I$ and pure strategy $s_i \in S_i$ let $[s_i] \subseteq S_i$ denote its (payoff) *equivalence class*, that is, the set of pure strategies s_i' that are payoff equivalent with s_i.

Definition 4.2 *The* (purely) reduced *normal form representation* G^o *of a finite normal-form game* $G = \langle I, S, u \rangle$ *is the normal-form game* $G^o = \langle I, S^o, u^o \rangle$ *in which the pure strategies are the equivalence classes in* G*, and where* u^o *is the accordingly adapted payoff function;* $u^o([s_1], .., [s_n]) = u(s_1, .., s_n)\ \forall s \in S$*.*