SF2972 GAME THEORY

Infinite games

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1 Introduction

- So far, the course has been focused on **finite** games:
  - Normal-form games with a finite number of players, where each player has a finite set of pure strategies
  - Extensive-form games with a finite number of players, where each player has finitely many information sets and a finite choice set at each information set

- But we did in fact already consider certain **infinite** games — when we considered the mixed-strategy extension of games, and when we considered behavior strategies in extensive-form game
  - Each player's set of mixed strategies is then a unit simplex in a Euclidean (finite dimensional) space. Hence compact and convex.

And each player's payoff function is continuous, and linear in the player's own mixed strategy

– Each player's set of behavior strategies is a polyhedron, the finite Cartesian product of unit simplices (the set of local strategies at each information set). These are also compact and convex sets. However, although each player's payoff-function is continuous, it is in general not linear, not even quasi-concave, in the player's own behavior strategy

- In many applications of game theory, the games are **infinite**, in a number of distinct ways: players may have continuum strategy sets that are neither simplices nor polyhedra, strategy sets may even be infinite-dimensional, players may have countably infinitely many pure strategies or information sets, there may be infinitely many players (either countably many or a continuum)

- We will today consider infinite (normal-form and extensive-form) games

2 Infinite normal-form games

Recap:

Definition 2.1 *A* **normal-form game** *is a triplet* $G = \langle I, S, u \rangle$, *where*

(i) I *is the (non-empty) set of* **players**

(ii) $S = \times_{i \in I} S_i$ *is the set of* **strategy profiles** $s = (s_i)_{i \in I}$ *with* S_i *denoting the non-empty* **strategy set** *of each player* $i \in I$

(iii) $u : S \to \mathbb{R}^{|I|}$ *is the* **combined payoff function**, *where* $u_i(s) \in \mathbb{R}$ *is the* **payoff** *to player* i *when strategy profile* s *is played*

- For each player i and strategy profile s, let

$$\beta_i(s) = \arg\max_{s_i' \in S_i} u_i\left(s_i', s_{-i}\right)$$

and let

$$\beta(s) = \times_{i \in I} \beta_i(s)$$

- This defines a correspondence $\beta : S \rightrightarrows S$ if it is non-empty valued (which is not always the case)

Definition 2.2 *A strategy profile* $s^* \in S$ *is a* **Nash equilibrium** *of* G *if* $s^* \in \beta(s^*)$.

- G is *finite* if the set S is finite

- G is *Euclidean* if the set S is a subset of some Euclidean space (finitely many players, each player's strategy set a subset of some Euclidean space)

- Special cases: all finite games, and also their mixed-strategy extensions

Example 2.1 *Reconsider the firm-worker example, but now with a continuum range of wages. The firm owner, player 1, offers a wage* $w \in W = [0, 100]$ *to a worker, player 2. The worker has a binary choice, to either accept the offer,* $y = 1$*, or reject it,* $y = 0$*. In the first case, the owner makes a profit of* $100 - w$ *and the worker earns income* w*, while in the second case the firm earns zero profit and the worker earns her reservation wage* $w_0 \in (0, 100)$*.*

The worker's strategy set is infinite-dimensional, the set of functions $f : W \to \{0, 1\}$*. Hence, this is* **not** *a Euclidean game.*

We obtain a Euclidean game if we restrict the worker to cut-off strategies, *step functions* f *that jump up from 0 (rejection) to 1 (acceptance) at some critical wage* $x \in W$*. We may then view* G *as a game on the square* $S = [0, 100]^2$*, in which the firm owner picks an offer* $w \in [0, 100]$*, the* *worker an acceptance wage $x \in [0, 100]$, and the payoff functions are π_1 and π_2, where*

$$\pi_1(w, x) = \begin{cases} 100 - w & \textit{if } w \geq x \\ 0 & \textit{otherwise} \end{cases}$$

and

$$\pi_2(w, x) = \begin{cases} u(w) & \textit{if } w \geq x \\ u(w_0) & \textit{otherwise} \end{cases}.$$

where u is the worker's Bernoulli function. What is the set of **Nash equilibrium** *wages? Viewed as an infinite extensive-form game, what are the subgames? What is the set of* **subgame-perfect equilibrium** *wages?*

- Do all Euclidean games have Nash equilibria? No, but:

Theorem 2.1 *Let $G = \langle I, S, u \rangle$ be a Euclidean game in which each strategy set S_i is non-empty, compact and convex, and each payoff function $u_i : S \to \mathbb{R}$ is continuous. If each payoff function u_i is quasi-concave in $s_i \in S_i$ (for any given $s_{-i} \in S_{-i}$) then G has at least one Nash equilibrium.*

Proof: By Weierstrass' Maximum Theorem, $\beta_i(s)$ is non-empty and compact for every $s \in S$. By quasi-concavity and the convexity of S_i, $\beta_i(s)$ is convex. By Berge's Maximum Theorem, each correspondence β_i is upper hemi-continuous. The combined best-reply correspondence $\beta : S \rightrightarrows S$, defined by $\beta(s) = \times_{i \in I} \beta_i(s)$, inherits these properties. Thus all conditions in Kakutani's Fixed-Point Theorem are met, so β has at least one fixed point. **Q.E.D.**

- Does the theorem apply to the firm-worker example?

- The mixed-strategy extension $\tilde{G} = \langle I, \boxdot(S), \tilde{\pi} \rangle$ of any finite game $G = \langle I, S, \pi \rangle$ meets the conditions of the above existence theorem

- More generally, by representing mixed strategies by Borel probability measures μ_i on the pure-strategy sets S_i, any Euclidean game $G = \langle I, S, \pi \rangle$ with compact strategy sets and continuous payoff functions has a mixed-strategy extension $\tilde{G} = \langle I, \boxdot(S), \tilde{\pi} \rangle$, where

$$\tilde{\pi}_i(\mu) = \int_{S_1} \ldots \int_{S_n} \pi_i(s)\, d\mu_1(s_1) \ldots d\mu_n(s_n)$$

Proposition 2.2 (Glicksberg, 1952) *Let $G = \langle I, S, \pi \rangle$ be a Euclidean game in which each strategy set S_i is non-empty and compact, and where each payoff function $\pi_i : S \to \mathbb{R}$ is continuous. Its mixed-strategy extension, the game $\tilde{G} = \langle I, \boxdot(S), \tilde{\pi} \rangle$, has at least one Nash equilibrium.*

3 Examples

Consider n decision-makers i who each has to choose an "action" s_i in some closed and bounded interval, say, $S_i = [0, b_i]$ for $b_i > 0$. Each decision-maker i obtains utility or profit as the difference between a "benefit" and a "cost" that may depend on everybody's actions:

$$\pi_i(s) = B_i(s) - C_i(s)$$

- If $B_i, C_i : S \to \mathbb{R}$ are continuous, then Theorem 2.2 applies, so any such game has at least one NE in pure or mixed strategies

- If, moreover, each B_i is concave in s_i and each C_i convex in s_i, for any given subprofile s_{-i} of others' actions, then Theorem 2.1 applies, so any such game has at least one NE in pure strategies

- Applications abound!

3.1 Cournot competition

[Cournot, 1838]

Firms competing in a product market by way of choosing their individual outputs, with market-clearing prices

- Continuous payoff functions of the form

$$\pi_i (q_1, ..., q_n) = P \left(\sum_{j=1}^{n} q_j \right) \cdot q_i - C_i (q_i)$$

where $q_i \geq 0$ is the output (or supply) of firm i, $C_i (q_i)$ is its production cost, and $p = P(Q)$ is the market price when aggregate output is Q

Example of Cournot duopoly

1. Two identical firms, simultaneously choosing outputs $q_1, q_2 \in [0, 100]$

2. No fixed cost of production, constant marginal cost, $C_i(q_i) = c \cdot q_i$ for some $c < 100$

3. Demand at any price $p \in [0, 100]$:

$$D(p) = 100 - p$$

4. Market-clearing price at any aggregate output $Q \geq 0$:

$$P(Q) = 100 - Q$$

5. Payoff=profit, and payoff functions

$$\pi_i(q_1, q_2) = (100 - q_1 - q_2) \cdot q_i - c \cdot q_i$$

6. We have defined a Euclidean game. Are the conditions of Theorem 2.1 met?

7. In class: show that this game has a unique NE, and that it is

$$q_1^* = q_2^* = \frac{100 - c}{3}$$

8. In class: Instead of equilibrium reasoning, use rationalizability!

3.2 Bertrand competition

[Bertrand, 1883]

Firms competing in a product market by way of choosing their individual prices, and producing what is demanded from them

- Discontinuous payoff functions of the form

$$\pi_i(p_1, ..., p_n) = D_i(p_1, ..., p_n) \cdot p_i - C_i(D_i(p_1, ..., p_n))$$

where $p_i \geq 0$ is the price posted by firm i, $C_i(q_i)$ is its production cost output q_i, and $D_i(p_1, ..., p_n)$ is the demand, and

$$D_i(p_1, ..., p_n) = \begin{cases} D(p_{\mathsf{min}}) & \text{if } p_i < \min_{j \neq i} p_j \\ D(p_{\mathsf{min}})/m & \text{if } p_i = \min_{j \neq i} p_j \\ 0 & \text{if } p_i > \min_{j \neq i} p_j \end{cases}$$

for some continuos (demand) function D for the product in question, with $m > 1$ denoting the number of firms who quote the lowest price

Example of Bertrand duopoly

1. Two identical firms, simultaneously choosing prices $p_1, p_2 \in [0, 100]$

2. No fixed cost of production, constant marginal cost, $C_i(q_i) = c \cdot q_i$ for some $c < 100$

3. Demand at any price $p \in [0, 100]$:

$$D(p) = 100 - p$$

4. Payoff=profit, and payoff functions (for $i = 1, 2$ and $j \neq i$):

$$\pi_i(p_1, p_2) = \begin{cases} (100 - p)(p - c) & \text{if } p_i < p_j \\ (100 - p)(p - c)/2 & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

5. We have defined a Euclidean game. Are the conditions of Theorem 2.1 met?

6. In class: show that this game has a unique NE, and that it is

$$p_1^* = p_2^* = c$$

7. Note that the unique equilibrium strategies are weakly dominated

8. Assume a smallest monetary unit in which prices have to be expressed. This defines a finite game. Hence, it has at least one undominated NE (in pure or mixed strategies). Why? Show that (if the monetary unit is small) there are two NE in pure strategies, one dominated, the other perfect.

3.3 Cooperation and public goods

Individuals who all enjoy a public good, and to which each individual makes a voluntary individual contribution

- Continuous payoff functions of the form

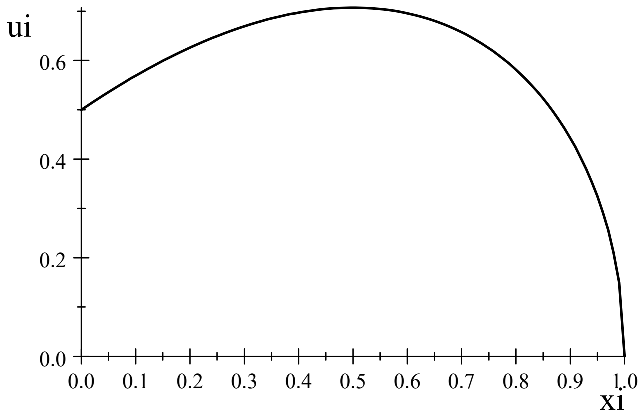
$$\pi_i(x_1, ..., x_n) = F\left(\sum_{j=1}^{n} x_j\right) - C_i(x_i)$$

where $x_i \geq 0$ of everybody' contribution, F is a continuous function that represents production of the public good, here taken to depend on the sum of all individual contributions, and $C_i(x_i)$ is the cost for individual i to contribute x_i

Example of public goods game

1. Two individuals, $i = 1, 2$. Each individual has to choose an effort level $x_i \in [0, 1]$, resulting in provision $x_1 + x_2$ of a public good, and in utilities

$$u_i(x_1, x_2) = (x_1 + x_2) \cdot (1 - x_i)^{1/2}.$$



2. Are the conditions of Theorem 2.1met?

3. Find the best-reply correspondence of each player. For player 1, we have

$$\frac{\partial u_1(x)}{\partial x_1} = (1 - x_i)^{1/2} - \frac{x_1 + x_2}{2\,(1 - x_i)^{1/2}}$$

4. The necessary first-order condition (FOC) for x_1 to be optimal for player 1 then is

$$x_1 = \frac{2 - x_2}{3}$$

5. Doing likewise for player 2, we find the unique NE $x_1^* = x_2^* = 1/2$

6. [Homework:] Now suppose individual 1 has to select x_1 before individual 2 selects x_2, and that individual 2 observes x_1 before selecting x_2.

Specify this as a normal-form game. What is the strategy set of player 1, player 2? Is it Euclidean?

(a) Find the unique subgame perfect equilibrium (in pure strategies). Does individual 1 now make more or less effort than in the simultaneous-move game?

(b) Find a Nash equilibrium that is not subgame perfect. Explain, in terms of "threats" and/or "promises," whether this Nash equilibrium is plausible or not

(c) Find the common effort level that would maximize the sum of the individuals' utility; the *socially optimal* effort level

3.4 Horizontal differentiation and competition

[Hotelling, 1929]

Two players, continuum strategy sets, discontinuous payoff functions

1. Consider two ice-cream vendors, A and B, who sell the same ice-cream to a continuum of consumers, spread out on a beach. Let f be the population density and F the cumulative population distribution function. (Normalize the total population to unity.)

2. The vendors have no fixed costs, each vendor has a unit cost of $c < 1$ euros per ice-cream, and they have to sell each ice-cream at the same fixed price p

3. Each vendor has to choose a location, x_A and x_B, respectively

4. Each consumer buys exactly one ice-cream, from the nearest vendor. If the two vendors stand at the same location, all consumers split even between them.

   (a) Here $\tau > 0$ is the transportation cost (or disutility or inconvenience) for the consumer of going to the vendor in question

5. Suppose the consumer are uniformly spread out on the unit interval.

   (a) If you were a social planner who could decide at what locations, x_A and x_B the ice-cream vendors can put up their stands, what would you then decide if the goal was to minimize consumers' total distance to the nearest vendor?

(b) If the two vendors are free to choose their locations, and they would do so simultaneously, where would they set up their stands? Write this up as a normal form game. Is it Euclidean? Are payoff functions continuous? Do best replies always exist? Does a NE exist? Do they earn more than when their locations were regulated? Are consumers better or worse off?

6. Solve for NE, as in 5 (b), but for an arbitrary population density f on the real line

7. Consider an alternative interpretation in terms of policy positions and competition for votes

8. In the case of ice-cream vendors: what if they can set their prices themselves? [Hotelling, 1929, d'Apremont et al (1979)]

3.5 The Rubinstein-Ståhl bargaining model

[Ståhl, 1972, Rubinstein, 1982]

Two players, infinite-dimensional strategy sets, countably infinitely many (singleton) information sets

- Informally in class

1. Two parties bargain over how to divide a unit of surplus (a "cake"). If one party gets the share $x \in [0, 1]$, the the other gets $y = 1 - x$

2. Both parties are selfish

3. In each round $t = 0, 1, 2, ...$ one party gives an offer $x_t \in [0, 1]$ to the other, which the other party can accept or reject

   (a) If accept they split the cake according to the agreement, $1 - x_t$ and x_t, and the game ends

   (b) If reject, the game goes to the next round, $t + 1$, and the rejector in round t gives an offer $x_{t+1} \in [0, 1]$ to the other party

4. Can you write this up as an extensive-form game? As a normal-form game? How specify payoff functions?

3.6 Repeated games

The same simultaneous-move game played in time periods $t = 0, 1, 2, \ldots.$

- Another lecture!