SF2972 GAME THEORY
Evolutionary game theory

Jörgen Weibull

February 2017

1 Economic theory and ”as if” rationality

- The rationalistic paradigm in economics: Savage rationality [Leonard Savage: *The Foundations of Statistics*, 1954]
  - Each economic agent's behavior derived from maximization of some goal function (utility, profit), under given constraints and information

- The ”as if” defence by Milton Friedman (1953): *The methodology of positive economics*
  - Firms that do not take profit-maximizing actions are selected against in the market

- But is this claim right? Under perfect competition? Under imperfect competition?

- Evolutionary theorizing older than Darwin: De Mandeville, Malthus, even Aristote

- Darwin: exogenous environment - “perfect competition”

- Maynard Smith: endogenous environment - “imperfect competition”

- *Evolutionary game theory* provides concepts and methods to rigorously explore Nash’s mass-action interpretation

- The “folk theorem” of evolutionary game theory:
  - If a stationary population distribution is *dynamically stable*, then it constitutes a Nash equilibrium
  - If the population process *converges* from an interior initial state, then the limit distribution is a Nash equilibrium
  - If the population process starts from an interior state, then iteratively strictly dominated strategies will be asymptotically wiped out
- Natural selection among behaviors may lead to apparent game-theoretic rationality, such as rationalizability and equilibrium play

1.1 Evolutionary game theory

- Evolutionary process =

  = mutation process + selection process

- The unit of selection: usually strategies ("strategy evolution"), sometimes utility functions ("preference evolution")

1. **Evolutionary stability**: focus on robustness to mutations

2. **Replicator dynamic**: focus on selection. [Robustness to mutations by way of dynamic stability]

3. **Stochastic stability**: both selection and mutations

1.2 Evolutionary stability of strategies

- ESS = *evolutionarily stable strategy* [Maynard Smith and Price (1973)]

  - "a strategy that 'cannot be overturned', once it has become the 'convention' in a population

Maynard Smith and Price: Consider a large population of individuals who are recurrently and (uniformly) randomly matched in pairs to play a finite and symmetric game

1. Initially, all individuals use the same pure or mixed strategy, x, the *incumbent*, or *resident*, strategy

2. Suddenly, a small population share $\varepsilon > 0$ switch to another pure or mixed strategy, y, the *mutant* strategy

- If the residents on average *do better* than the mutants, then x is *evolutionarily stable against* y,

- A strategy x is *evolutionarily stable* if it is evolutionarily stable against *all* mutants $y \neq x$

2 Evolutionary stability analysis

2.1 Domain

- *Symmetric finite two-player games in normal form*

Definition 2.1 *A two-player game* $G = \langle \{1, 2\}, S, u \rangle$ *is* **symmetric** *if* $S_1 = S_2$ *and* $u_2(h, k) = u_1(k, h)$ $\forall h, k \in S_1 = S_2$.

- With payoff bimatrix (A, B), where $A = (a_{hk})$, $B = (b_{hk})$, the game is symmetric iff $B = A^T$

Example 2.1 (Prisoners' dilemma)

$$\begin{array}{ccc} & C & D \\ C & 3,3 & 0,4 \\ D & 4,0 & 1,1 \end{array}$$

$$A = \begin{pmatrix} 3 & 0 \\ 4 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix}$$

Symmetric since $B = A^T$.

Example 2.2 (Matching Pennies)

$$\begin{array}{ccc} & H & T \\ H & 1,-1 & -1,1 \\ T & -1,1 & 1,-1 \end{array}$$

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Here $B^T \neq A$. Not a symmetric game.

- Thus, matching pennies games fall outside the domain of evolutionary stability analysis

Example 2.3 (Coordination game) *Payoff bimatrix:*

$$\begin{array}{ccc} & L & R \\ L & 2,2 & 0,0 \\ R & 0,0 & 1,1 \end{array}$$

$$A = B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

A **doubly** *symmetric game:* $B = A^T = A$, *an example of a* **potential game** *[Rosenthal (1974), Monderer and Shapley (1996)]*

2.2 Notation

- Write S for the (common) strategy set, $S = S_1 = S_2$

- Write Δ for $\Delta(S)$, the (common)mixed-strategy simplex:

$$\Delta = \{x \in \mathbb{R}_+^m : \sum_{i \in S} x_i = 1\}$$

- Write the payoff to any strategy $x \in \Delta$, when used against any strategy $y \in \Delta$ as

$$\pi(x, y) = x \cdot Ay$$

Note that the first argument, x, is *own* strategy, and the second argument, y, the *other* party's strategy

- Mixed best replies to $x \in \Delta$:

$$\beta^*(x) = \{x^* \in \Delta : \pi(x^*, x) \geq \pi\left(x', x\right) \ \forall x' \in \Delta\}$$

- This defines a correspondence from Δ to itself: $\beta^* : \Delta \rightrightarrows \Delta$

  [$\neq$ usual BR correspondence, which maps $\boxdot = \Delta^2$ to Δ]

- Let

$$\Delta^{NE} = \{x \in \Delta : x \in \beta^*(x)\}$$

- Note $x \in \Delta^{NE} \Leftrightarrow (x, x)$ is a symmetric NE

Proposition 2.1 $\Delta^{NE} \neq \varnothing$.

Proof: Application of Kakutani's Fixed-Point Theorem.

2.3 Definition of ESS

Definition 2.2 *$x \in \Delta$ is an* **evolutionarily stable strategy (ESS)** *if for every strategy $y \neq x$ $\exists$ $\bar{\varepsilon}_y \in (0, 1)$ such that for all $\varepsilon \in (0, \bar{\varepsilon}_y)$:*

$$\pi[x, \varepsilon y + (1-\varepsilon)x] > \pi[y, \varepsilon y + (1-\varepsilon)x].$$

- "Post-entry population mixture":

  $$p = \varepsilon y + (1-\varepsilon)x \in \Delta$$

  a convex combination of x and y, a point on the straight line between them

- Note that $\bar{\varepsilon}_y$ may be "tailored" for the particular mutant y at hand

- Let $\Delta^{ESS} \subset \Delta$ denote the set of ESSs

- Note that an ESS has to be a *best* reply to itself: if $x \in \Delta^{ESS}$ then $\pi(y, x) \leq \pi(x, x)$ for all $y \in \Delta$

- Hence $\Delta^{ESS} \subset \Delta^{NE}$

- Note also that an ESS has to be a *better* reply to its alternative best replies than they are to themselves: if $x \in \Delta^{ESS}$, $y \in \beta^*(x)$ and $y \neq x$, then $\pi(x, y) > \pi(y, y)$

Proposition 2.2 *$x \in \Delta^{ESS}$ if and only if for all $y \neq x$:*

$$\pi(y, x) \leq \pi(x, x)$$

and

$$\pi(y, x) = \pi(x, x) \Rightarrow \pi(y, y) < \pi(x, y)$$

- $\Rightarrow$ the strategy used in any strict symmetric NE is an ESS

2.4 Examples

2.4.1 Prisoner's dilemma

$$\begin{array}{ccc} & C & D \\ C & 3,3 & 0,4 \\ D & 4,0 & 2,2 \end{array}$$

$$\Delta^{ESS} = \Delta^{NE} = \{D\}$$

2.4.2 Coordination game

	L	R
L	$2,2$	$0,0$
R	$0,0$	$1,1$

$$\Delta^{NE} = \left\{ L, R, \frac{1}{3}L + \frac{2}{3}R \right\}$$

$$\Delta^{ESS} = \{L, R\}$$

The mixed NE is *perfect* and even *proper*, but not evolutionarily stable!

2.4.3 Hawk-dove game

- The original example of Maynard Smith and Price (1972)

- Start-up two-partner businesses, or pairs of students assigned to write an essay together

- Each partner has to choose between *work* ("contribute") and *shirk* ("free-ride"):

$$\begin{array}{ccc} & W & S \\ W & 3,3 & 0,4 \\ S & 4,0 & -1,-1 \end{array}$$

- Symmetric game (but **not** a Prisoners' Dilemma)

- Consider a large pool of individuals and random matching

1. Unique symmetric NE: randomize uniformly, $x^* = (1/2, 1/2)$, $\Delta^{NE} = \{x^*\}$. Hence $\Delta^{ESS} \subset \{x^*\}$

2. x^* an ESS iff

$$\pi(x^*, y) > \pi(y, y) \qquad \forall y \neq x^*$$

3. Equivalently:

$$\frac{1}{2}\left[3y_1 + 4y_1 - (1 - y_1)\right] > 3y_1^2 + 4y_1\left(1 - y_1\right) - \left(1 - y_1\right)^2$$

or

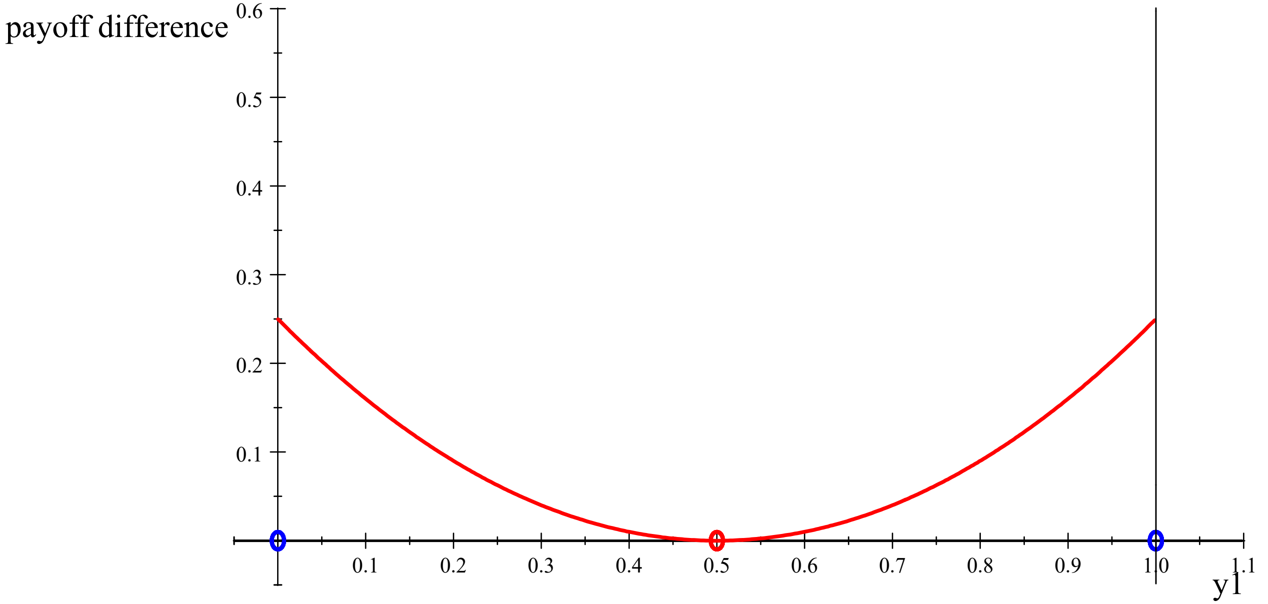
$$8y_1 - 1 > -4y_1^2 + 12y_1 - 2$$

or

$$4\left(y_1 - \frac{1}{2}\right)^2 > 0$$

4. True, hence x^* is an ESS!

- Payoff difference $\pi(x^*, y) - \pi(y, y)$:



- Some games have no ESS. For instance, when all payoffs are the same. But also in more interesting games such as

Example 2.4 (Rock-scissors-paper) *Rock beats Scissors, Scissors beat Paper, and Paper beats Rock:*

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Unique Nash equilibrium: $x^* = (1/3, 1/3, 1/3)$*. All pure strategies are best replies and do just as well against themselves as* x^* *does against them:* $\Delta^{ESS} = \varnothing$*.*

3 Relations to non-cooperative solution concepts

- Evolutionary stability not only implies that the strategy is a best reply to itself, it also implies that the strategy is not weakly dominated:

Proposition 3.1 $x \in \Delta^{ESS} \Rightarrow x$ *undominated.*

Corollary 3.2 *Hence:* $x \in \Delta^{ESS} \Rightarrow (x, x)$ *is a perfect equilibrium.*

- One can prove that ESS even implies properness:

Proposition 3.3 (van Damme, 1987) $x \in \Delta^{ESS} \Rightarrow (x, x)$ *is a proper equilibrium.*

- Hence, every ESS induces a (realization-equivalent) sequential equilibrium in every EF-game with the given NF!

- All roads lead to Rome ...

THE END