KTH Teknikvetenskap

SF2972 Game Theory
Written Exam with Solutions
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Part A – Classical Game Theory

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1. Finite normal-form games.

(a) What are N, S and u in the definition of a *finite normal-form* (or, equivalently, *strategic-form*) *game* $G = \langle N, S, u \rangle$? **[1 pt]**

(b) Give the definition of a *strictly dominated* (pure or mixed) strategy in such a game. **[1 pt]**

(c) Give the definition of a *Nash equilibrium* (in pure or mixed strategies) in such a game. **[1 pt]**

(d) For finite and symmetric two-player games G: give the definition of an *evolutionarily stable* (pure or mixed) strategy. **[1 pt]**

Solution *See Osborne-Rubinstein and lecture slides.*

2. Consider the two-player normal-form game G with payoff matrix

$$\begin{array}{cccc} & a & b & c \\ a & 6,6 & 0,0 & 0,7 \\ b & 0,0 & 1,1 & 4,5 \\ c & 7,0 & 5,4 & 0,0 \end{array}.$$

(a) Find all pure strategies that are strictly dominated. **[1 pt]**

(b) Find all Nash equilibria in pure and/or mixed strategies. **[2 pts]**

(c) Find all evolutionarily stable strategies. **[1 pt]**

Solution

(a) There is only one pure strategy that is strictly dominated, namely a, which is dominated by any $(0, \varepsilon, 1-\varepsilon)$ for $0 < \varepsilon < \frac{1}{7}$.

(b) The Nash equilibria are (c, b), (b, c), and $\big((0, \frac{1}{2}, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{2})\big)$.

(c) There is only one evolutionary stable strategy, namely $(0, \frac{1}{2}, \frac{1}{2})$.

3. Two individuals, Al and Beth, contribute to a public good (say, a clean shared office) by making individual efforts $x \geq 0$ and $y \geq 0$. Individual utilities are given by

$$u_1(x,y) = (x+y)^a - ax^2 \quad \text{and} \quad u_2(x,y) = (x+y)^a - ay^2$$

for some $a \in (0,1)$. Each individual strives to maximize his or her utility.

(a) Game A: Suppose both effort levels are chosen simultaneously. Write up the normal form $G_A = \langle N, S, u \rangle$ of this game. Prove that there exists a unique Nash equilibrium in pure-strategies, and identify this equilibrium. **[2 pts]**

(b) Game B: Suppose that Al first chooses his effort level, x, and that this is observed by Beth, who then chooses her effort level, y. Write up the normal form $G_B = \langle N, S, u \rangle$ of this sequential game. **[1 pt]**

(c) Are the Nash equilibrium efforts in Game A, x^* and y^*, taken in any Nash equilibrium in Game B? **[2 pts]**

Solution (a) Each payoff function is differentiable on $\mathbb{R}^2_{++}$. F.O.C. both necessary and sufficient for NE, and has the unique solution $(x^*, y^*) = (1/2, 1/2)$.

(b) $S_1 = \mathbb{R}_+$ and S_2 is the set of functions $f : \mathbb{R}_+ \to \mathbb{R}_+$. Payoff functions: $\pi_i(x, f) = u_i(x, f(x))$.

(c) Let f^* be defined by

$$f^*(x) = \begin{cases} 0 & \text{if } x < 1/2, \\ 1/2 & \text{if } x \geq 1/2. \end{cases}$$

Then (x^*, f^*) is a NE as requested.

4. A child's action a (from a nonempty, finite set A) affects both her own private income $c(a)$ and her parents' income $p(a)$; for all $a \in A$ we have $0 \leq c(a) < p(a)$. The child is selfish: she cares only about the amount of money $c(a)$ she has. Her loving parents care both about how much money they have and how much their child has. Specifically, model the parents as a single player whose payoff equals the smaller of the amount of money the parents have and the amount of money the child has. The parents may transfer money to the child.

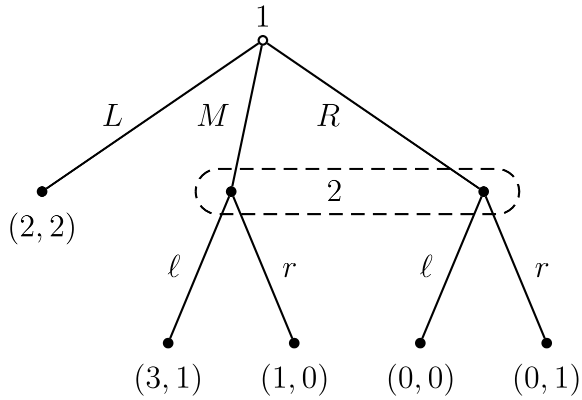
First the child takes an action $a \in A$. Then the parents observe the action and decide how much money $x \in [0, p(a)]$ to transfer to the child. The game ends with payoffs $c(a) + x$ to the child and $\min\{c(a) + x, p(a) - x\}$ to the parents.

Show that in a subgame perfect equilibrium the child takes an action that maximizes the sum of her private income and her parents' income. Not so selfish after all! **[4 pts]**

Solution

- ⊠ In the subgame after action $a \in A$, the parents maximize $\min\{c(a)+x, p(a)-x\}$ over $x \in [0, p(a)]$. This is done by choosing x such that $c(a) + x = p(a) - x$, i.e., by $x^*(a) = \frac{1}{2}(p(a) - c(a))$.
- ⊠ Anticipating this, the child knows that action $a \in A$ leads to transfer $x^*(a)$ and consequently payoff $c(a) + x^*(a) = \frac{1}{2}(c(a) + p(a))$. Maximizing this is equivalent (just multiply with 2) with maximizing $c(a) + p(a)$.

5. Consider the game below.



(a) Find the corresponding strategic form game. What is the outcome of iterated elimination of weakly dominated strategies (IEWDS)?

(b) Find all sequential equilibria. Compare the outcomes under (a) and (b): which do you find most reasonable?

[8 pts]

Solution (a) The corresponding strategic game is

	ℓ	r
L	$2,2$	$2,2$
M	$3,1$	$1,0$
R	$0,0$	$0,1$

IEWDS consecutively eliminates R (*strictly* dominated), r, and L: only Nash equilibrium (M,ℓ) survives.

(b) Ignoring player 1's trivial assessment in the singleton information set at the initial node, we can denote an assessment by

$$\begin{aligned}(\beta,\mu) &= (\beta_1,\beta_2,\mu)\\ &= (\underbrace{(\beta_1(\emptyset)(L),\beta_1(\emptyset)(M),\beta_1(\emptyset)(R))}_{\text{beh. str. of pl. 1 over }\{L,M,R\}},\underbrace{(\beta_2(\{M,R\})(\ell),\beta_2(\{M,R\})(r))}_{\text{beh. str. of pl. 2 over }\{\ell,r\}},\underbrace{(\mu(\{M,R\})(M),\mu(\{M,R\})(R))}_{\text{beliefs over }\{M,R\}}).\end{aligned}$$

Since R is strictly dominated, sequential rationality requires that it is played with zero probability in a sequential equilibrium:

$$\beta_1(\emptyset)(R) = 0. \tag{1}$$

Distinguish two cases:

Case 1: sequential equilibria with $\beta_1(\emptyset)(M) > 0$:

- ⊠ Consistency requires that in information set $\{M,R\}$, which is reached with positive probability, the beliefs are derived from $\beta_1(\emptyset)$ using Bayes' rule. Together with (1), this gives $\mu(\{M,R\})(M) = 1, \mu(\{M,R\})(R) = 0$: player 2 believes to be in node M with probability 1.
- ⊠ Player 2's unique best response to this belief is to choose ℓ: $\beta_2(\{M,R\})(\ell) = 1$.
- ⊠ Consequently, player 1 prefers M (giving payoff 3) to L (payoff 2): $\beta_1(\emptyset)(M) = 1$.
- ⊠ We found a candidate sequential equilibrium:

$$(\beta_1,\beta_2,\mu) = ((0,1,0),(1,0),(1,0)),$$

corresponding to the equilibrium in the strategic game that survives iterated elimination.
- ⊠ To verify consistency, notice that (β_1,β_2,μ) is the limit of

$$\big((\varepsilon, 1-2\varepsilon, \varepsilon), (1-\varepsilon,\varepsilon), (\tfrac{1-2\varepsilon}{1-\varepsilon}, \tfrac{\varepsilon}{1-\varepsilon})\big), \qquad 0<\varepsilon\to 0$$

Case 2: sequential equilibria with $\beta_1(\emptyset)(M) = 0$:

- ⊠ Together with (1), this implies that $\beta_1(\emptyset)(L) = 1$. For notational convenience, denote player 2's strategy β_2 by $(p, 1-p)$ and the belief μ over $\{M, R\}$ by $(q, 1-q)$.
- ⊠ Sequential rationality requires that L is a best response. Player 1 won't play R (see (1)) and M gives expected payoff $3p + (1-p) = 2p + 1$, so L is a best response as long as $2p + 1 \leq 2$, i.e., as long as $p \in [0, 1/2]$.
- ⊠ But is every such p sequentially rational? If $0 < p \leq 1/2$, both actions ℓ and r are chosen with positive probability and therefore have to be best responses to 2's beliefs in his information set. This is true only if $q = 1/2$. If $p = 0$, only action r is chosen with positive probability. For this to be a best response to 2's beliefs in his information set, we need that $q \in [0, 1/2]$.
- ⊠ We found the following candidates for sequential equilibria:

$$\{(\beta_1, \beta_2, \mu) = ((1,0,0), (p, 1-p), (1/2, 1/2)) \mid 0 < p \leq 1/2\},$$

and

$$\{(\beta_1, \beta_2, \mu) = ((1,0,0), (0,1), (q, 1-q)) \mid 0 \leq q \leq 1/2\}.$$

- ⊠ It remains to verify that these assessments are consistent. For the first class of equilibria, let $0 < p \leq 1/2$. The assessment is the limit of

$$\big((1-2\varepsilon, \varepsilon, \varepsilon), (p, 1-p), (1/2, 1/2)\big), \qquad 0 < \varepsilon \to 0.$$

For the second class of equilibria, let $0 \leq q \leq 1/2$. The assessment is the limit of

$$\big((1-\varepsilon, (q+\varepsilon)\varepsilon, (1-q-\varepsilon)\varepsilon), (\varepsilon, 1-\varepsilon), (q+\varepsilon, 1-q-\varepsilon)\big), \qquad 0 < \varepsilon \to 0.$$

Discussion: It can be argued that, since the other equilibria involve the play of iteratively dominated strategies, the equilibrium in case 1 is the most appealing.

Notice: Another approach to finding sequential equilibria is to first find the trembling-hand perfect equilibria of the agent strategic form. Here, this is relatively easy: the agent strategic form is the same as the strategic form in (a) and in two-player games, the perfect equilibria are the undominated (notice: not the iteratively undominated) Nash equilibria, i.e., the Nash equilibria of the 2×2 game with pure strategy space $\{L, M\} \times \{\ell, r\}$.

Part B – Combinatorial Game Theory

Jonas Sjöstrand

6. Consider the game of Nim with the additional rule that we are only allowed to remove one stick or a prime number of sticks.

(a) Find the Grundy value $g(P_n)$ of a pile P_n of n sticks, for $0 \le n \le 8$. **[2 pts]**

(b) State a conjecture for the value of $g(P_n)$ for general n. **[1 pt]**

(c) Prove your conjecture. **[1 pt]**

(d) Find a winning move from the three-pile position $(100, 50, 25)$. **[1 pt]**

Solution (a)

$$
\begin{aligned}
g(P_0) &= \operatorname{mex} \emptyset = 0 \\
g(P_1) &= \operatorname{mex}\{g(P_{1-1})\} = \operatorname{mex}\{0\} = 1 \\
g(P_2) &= \operatorname{mex}\{g(P_{2-1}), g(P_{2-2})\} = \operatorname{mex}\{1, 0\} = 2 \\
g(P_3) &= \operatorname{mex}\{g(P_{3-0}), g(P_{3-2}), g(P_{3-3})\} = \operatorname{mex}\{2, 1, 0\} = 3 \\
g(P_4) &= \operatorname{mex}\{g(P_{4-1}), g(P_{4-2}), g(P_{4-3})\} = \operatorname{mex}\{3, 2, 1\} = 0 \\
g(P_5) &= \operatorname{mex}\{g(P_{5-1}), g(P_{5-2}), g(P_{5-3}), g(P_{5-5})\} = \operatorname{mex}\{0, 3, 2, 0\} = 1 \\
g(P_6) &= \operatorname{mex}\{g(P_{6-1}), g(P_{6-2}), g(P_{6-3}), g(P_{6-5})\} = \operatorname{mex}\{1, 0, 3, 1\} = 2 \\
g(P_7) &= \operatorname{mex}\{g(P_{7-1}), g(P_{7-2}), g(P_{7-3}), g(P_{7-5}), g(P_{7-7})\} = \operatorname{mex}\{2, 1, 0, 2, 0\} = 3 \\
g(P_8) &= \operatorname{mex}\{g(P_{8-1}), g(P_{8-2}), g(P_{8-3}), g(P_{8-5}), g(P_{8-7})\} = \operatorname{mex}\{3, 2, 1, 3, 1\} = 0
\end{aligned}
$$

(b) Conjecture: $g(P_n)$ is the remainder when n is divided by four.

(c) Let $[n]$ denote the remainder when n is divided by four. Since $[n] = [n-k]$ only if k is a multiple of 4, the set $S_n = \{[n-p] : 1 \le p \le n,\ p \text{ is prime or } 1\}$ does not contain $[n]$. But all nonnegative integers smaller than $[n]$ belongs to the set $\{[n-1], [n-2], [n-3]\}$ if $n \ge 3$. Thus, $\operatorname{mex} S_n = [n]$ and the conjecture is true by induction.

(d) For example, taking one stick from the first pile is a winning move, since

$$g(P_{99} + P_{50} + P_{25}) = g(P_{99}) \oplus g(P_{50}) \oplus g(P_{25}) = 3 \oplus 2 \oplus 1 = 0.$$

7. Alice and Bob plays the following game: First, Alice chooses a continent, then Bob chooses a country in that continent, and finally Alice chooses a city in that country. However, they may only choose continents, countries and cities from the following list of ten of the greatest cities in the world.

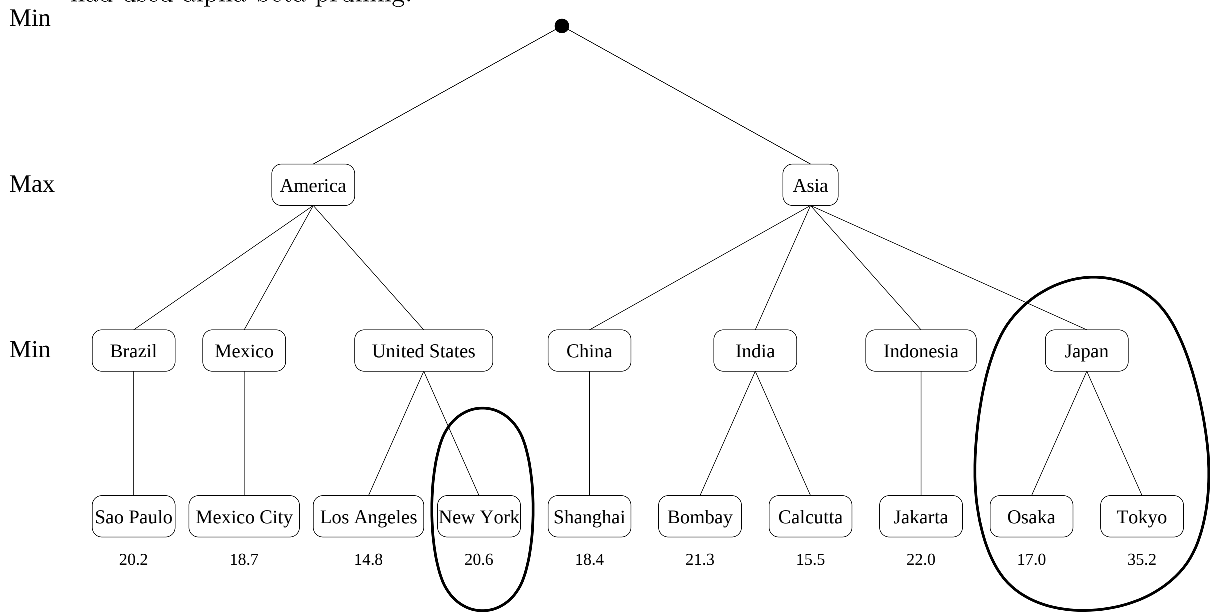
City	**Country**	**Continent**	**Population**/10^6
Tokyo	Japan	Asia	35.2
Jakarta	Indonesia	Asia	22.0
Bombay	India	Asia	21.3
New York	United States	America	20.6
São Paulo	Brazil	America	20.2
Mexico City	Mexico	America	18.7
Shanghai	China	Asia	18.4
Osaka	Japan	Asia	17.0
Calcutta	India	Asia	15.5
Los Angeles	United States	America	14.8

Alice wants to minimize the population of the chosen city, while Bob wants to maximize it. To compute the best strategy, Alice performs a complete minimax search on the game tree. When there are many possible choices, she decides to try them in alphabetical order.

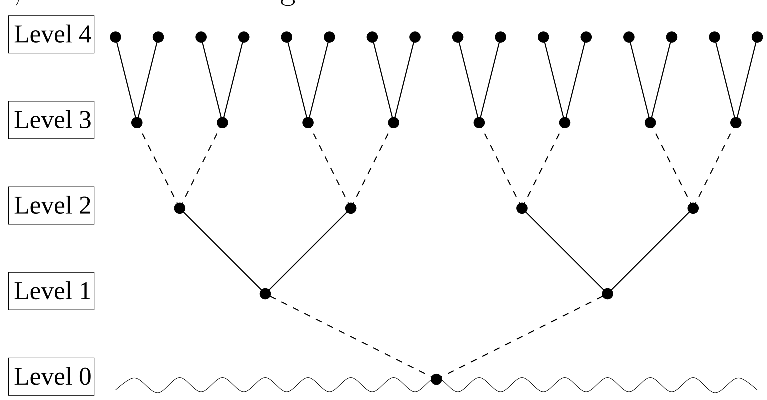
(a) Draw the complete game tree. **[2 pts]**

(b) Circle the parts of the game tree that would not have been explored if Alice had used alpha-beta pruning. **[3 pts]**

Solution Here is the complete game tree with those parts circled that would not have been explored if Alice had used alpha-beta pruning.



8. Let T_n be a binary tree of depth n with coloured edges such that edges between levels $i-1$ and i are blue if $n-i$ is even and red if $n-i$ is odd. For instance, T_4 looks like this, where the solid edges are blue and the dashed ones are red:



Let G_n denote the Blue-Red Hackenbush game played on T_n with the root connected to the ground.

(a) Compute the value of G_n for $0 \le n \le 7$. **[2 pts]**

(b) State a conjecture for the value of G_n for general n. **[1 pt]**

(c) Prove your conjecture. **[2 pts]**

Solution (a) Obviously, $G_0 = 0$, and for $n \geq 1$ we have the recurrence relation

$$G_n = \begin{cases} 2(-1 : G_{n-1}) \text{ if } n \text{ is even,} \\ 2(1 : G_{n-1}) \text{ if } n \text{ is odd.} \end{cases}$$

Thus,

$$\begin{aligned} G_0 &= 0 \\ G_1 &= 2(1 : G_0) = 2(1 : 0) = 2, \\ G_2 &= 2(-1 : G_1) = 2(-1 : 2) = -\tfrac{1}{2}, \\ G_3 &= 2(1 : G_2) = 2(1 : -\tfrac{1}{2}) = \tfrac{3}{2}, \\ G_4 &= 2(-1 : G_3) = 2(-1 : \tfrac{3}{2}) = -\tfrac{3}{4}, \\ G_5 &= 2(1 : G_4) = 2(1 : -\tfrac{3}{4}) = \tfrac{5}{4}, \\ G_6 &= 2(1 : G_5) = 2(-1 : \tfrac{5}{4}) = -\tfrac{7}{8}, \\ G_7 &= 2(1 : G_6) = 2(1 : -\tfrac{7}{8}) = \tfrac{9}{8}. \end{aligned}$$

(b) Conjecture: $G_{2m} = 2^{-m} - 1$ and $G_{2m+1} = 2^{-m} + 1$ for all nonnegative integers m.

(c) The conjecture is true for $m = 0$, so suppose $m \geq 1$ and argue by induction over m. By the recurrence relation above, we have $G_{2m} = 2(-1 : G_{2m-1})$ which, by the induction hypothesis, equals $2\big(-1 : (2^{-m+1}+1)\big)$. The sign-expansion of $2^{-m+1}+1 = 1+1-\sum_{k=1}^{m-1} 2^{-k}$ is $++(-)^{m-1}$; in other words, to reach $2^{-m+1}+1$ in Conway's number tree we should first go right twice and then left $m-1$ times. Thus, by definition of the colon operator, the sign-expansion of $-1 : (2^{-m+1}+1)$ is $-++(-)^{m-1}$. We conclude that

$$-1 : (2^{-m+1} + 1) = -1 + \tfrac{1}{2} + \tfrac{1}{4} - \sum_{k=3}^{m+1} 2^{-k} = 2^{-m-1} - \tfrac{1}{2}$$

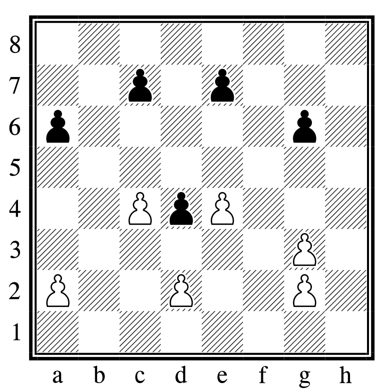
and hence $G_{2m} = 2^{-m} - 1$.

Now, again by the recurrence relation, $G_{2m+1} = 2(1 : G_{2m}) = 2\big(1 : (2^{-m} - 1)\big)$. The sign-expansion of $2^{-m} - 1 = -1 + \frac{1}{2} - \sum_{k=2}^{m} 2^{-k}$ is $-+(-)^{m-1}$, so the sign-expansion of $1 : (2^{-m} - 1)$ is $+-+(-)^{m-1}$. We conclude that

$$1 : (2^{-m} - 1) = 1 - \tfrac{1}{2} + \tfrac{1}{4} - \sum_{k=3}^{m+1} 2^{-k} = 2^{-m-1} - \tfrac{1}{2}$$

and hence $G_{2m+1} = 2^{-m} + 1$.

9. Consider a simplified version of chess where there are only pawns and where the normal play convention is adopted. Show that the following position is equal to 1 if Left is white.



(Recall that a pawn can move forward one square if that square is unoccupied. "Forward" means up in the diagram for white pawns and down for black pawns. A white pawn at rank 2 (that is, row 2) has also the option of moving two squares up provided both squares above the pawn are unoccupied. Analogously, a black pawn at rank 7 has the option of moving two squares down. No captures are possible in our example.)

You may use the identity $\{0 \mid \uparrow\} = \uparrow + \uparrow + *$, where $\uparrow = \{0 \mid *\}$, without proving it.

[5 pts]

Solution In chess, the columns of the board are called *files*. Let A be the game obtained by removing all pawns in files other than the a-file, and define C, D, E, and G analogously. Since a pawn cannot interact with a pawn in another file, the game position equals $A + C + D + E + G$.

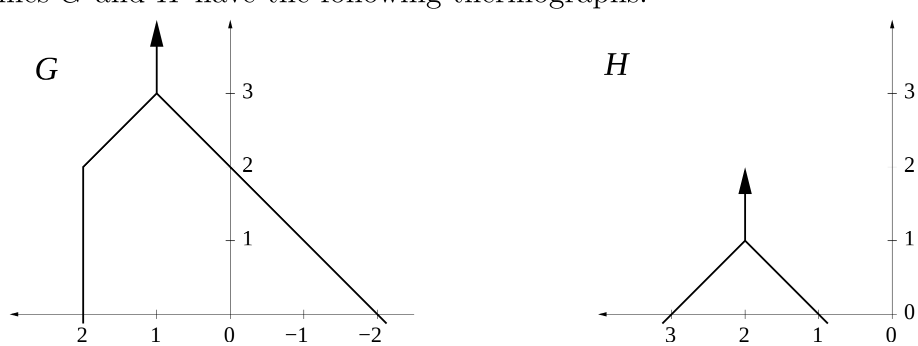
In the d-file, only one move is possible for either player, so $D = *$. In the c- and e-files, both players can move to $*$, but Right also has the option to move his pawn two squares forward which results in a zero game. So $C = E = \{*| \, 0, *\}$, but the right option $*$ is reversible through $0 \geq C$ (since left has no winning move from C) and thus $C = E = \downarrow = -\uparrow = \{*| \, 0\}$.

In the a-file, Right's only option is equal to $-C$ while left has two options: either he moves to a4 which results in $*$, or he moves to a3 which results in $\{*|*\} = 0$. So $A = \{*, 0 \mid \uparrow\}$, but the left option $*$ is reversible through $0 \leq A$ (since right has no winning move from A) and thus $A = \{0 \mid \uparrow\}$.

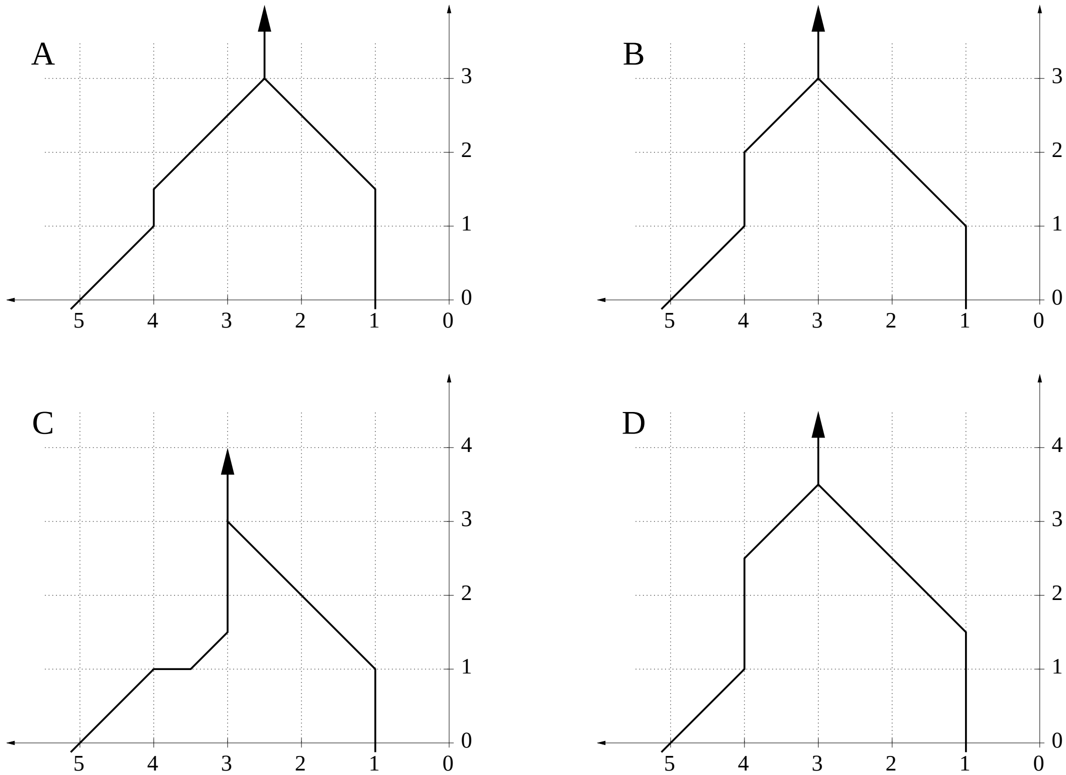
By the identity $\{0 \mid \uparrow\} = \uparrow + \uparrow + *$, we see that $A + C + D + E = 0$ so it remains only to show that $G = 1$. In the g-file, the players have one option each, let us call them G^L and G^R. From G^L, Right's only option is equal to 1 while Left has two options: either he moves to g5 which results in $\{0, 1|\} = 2$, or he moves to g3 which results in $\{1|0\} < 2$. Since dominated options can be removed, we have $G^L = \{2|1\}$. From G^R, Left's only option is equal to 1 while Right's only option is a zero game, so $G^R = \{1|0\}$.

We conclude that $G = \{G^L|G^R\} = \{2|1 \parallel 1|0\}$. Now, we can check that $G = 1$ for example by a strategic discussion (showing that $G - 1$ is a win for the second player) or by drawing the thermograph of G, or by showing that $G^L \lhd 1 \lhd G^R$ and using the simplicity theorem.

10. The games G and H have the following thermographs:



Which of the following graphs can possibly be the thermograph of $G+H$? (In each case, either give examples of G and H or prove that there are no such examples.) **[5 pts]**



Solution We see from the thermographs of G and H that $t(G) = 3$, $t(H) = 1$, $G_\infty = 1$, and $H_\infty = 2$. It follows that $t(G+H) \le \max\{t(G), t(H)\} = 3$ and $(G+H)_\infty = G_\infty + H_\infty = 3$. Diagram A or D cannot show the thermograph of $G+H$ since the mean value of A is not 3 and the temperature of D is greater than 3. Diagram C does not show a thermograph at all since there is a horizontal line segment in it.

The only diagram that possibly can show the thermograph of $G+H$ is B, and indeed it does if $G = \{6|2 \parallel -2\}$ and $H = \{3|1\}$. Then, $G+H = \{\{6|2\}+H, G+3 \parallel -2+H, G+1\}$ and, by the translation theorem,

$$\begin{aligned}
\{6|2\}+H &= \{6+\{3|1\}, \{6|2\}+3 \mid 2+\{3|1\}, \{6|2\}+1\} = \{\{9|7\}, \{9|5\} \mid \{5|3\}, \{7|3\}\},\\
G+3 &= \{\{6|2\}+3 \mid 1\} = \{9|5 \parallel 1\},\\
-2+H &= \{1 \mid -1\},\\
G+1 &= \{\{6|2\}+1 \mid -1\} = \{7|3 \parallel -1\}.
\end{aligned}$$

The options $\{9|5\}$ and $\{7|3\}$ of $\{\{9|7\}, \{9|5\} \mid \{5|3\}, \{7|3\}\}$ are dominated, so $\{6|2\}+H = \{9|7 \parallel 5|3\}$ and

$$G+H = \{\{9|7 \parallel 5|3\}, \{9|5 \parallel 1\} \mid \{1 \mid -1\}, \{7|3 \parallel -1\}\}.$$

But now the options $\{9|5 \parallel 1\}$ and $\{7|3 \parallel -1\}$ are dominated, so

$$G+H = \{\{9|7 \parallel 5|3\} \mid \{1 \mid -1\}\}.$$

Here are the thermographs of $G+H$ and its options in the same diagram: