KTH Teknikvetenskap

SF2972 Game Theory
Exam with Solutions
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PART A – CLASSICAL GAME THEORY

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1. Consider the following finite two-player game G, where player 1 chooses row and player 2 chooses column:

	a'	b'	c'	d'	e'
a	$-1,-1$	$-2,0$	$-2,-2$	$-1,0$	$0,-1$
b	$0,-1$	$0,0$	$0,0$	$0,0$	$0,0$
c	$0,0$	$0,0$	$3,3$	$6,0$	$6,0$
d	$0,0$	$0,0$	$0,6$	$4,4$	$7,0$

(a) Find all pure strategies that are *weakly* dominated (by a pure or mixed strategy). **[1.5 pts]**

(b) Find all pure strategies that are *strictly* dominated (by a pure or mixed strategy). **[1.5 pts]**

(c) Find all pure-strategy *Nash* equilibria. **[1.5 pts]**

(d) Find all pure-strategy *perfect* equilibria. **[1 pt]**

(e) Find all pure-strategy *proper* equilibria **[1 pt]**

Solution

(a) a, a', b, b' and e'

(b) a and a'

(c) (b, b') and (c, c')

(d) (c, c')

(e) (c, c')

2. There are $n \geq 1$ individuals who together maintain a forest. Each individual i makes effort $x_i \geq 0$, resulting in utility

$$u_i(x_1, ..., x_n) = \frac{100y}{1+y} - x_i$$

for the individual, where $y = x_1 + \cdots + x_n$ is the total amount of efforts to maintain the forest. Hence, everybody gets more utility from the forest the better it is maintained. All individuals choose their efforts simultaneously. Each individual strives to maximize his or her utility.

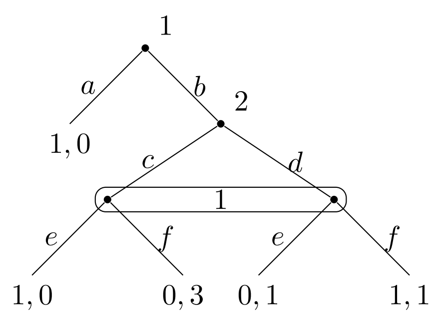
(a) Show that this game has exactly one symmetric Nash equilibrium (in pure strategies), that is, an equilibrium in which all individuals make the same effort, x^*. Find x^* and the associated total amount of effort, $y^* = nx^*$. **[2 pts]**

(b) Suppose that the individuals can pre-commit to a common effort level, the same for all. Let $\hat{x}$ be the common effort level that maximizes the sum of all individuals' utilities. Find $\hat{x}$ and the associated total amount of effort, $\hat{y} = n\hat{x}$. Compare these (socially optimal) levels with those in Nash equilibrium. Is there a difference when $n = 1$? When $n > 1$? Explain! **[1.5 pts]**

Solution (a) Strictly concave payoff functions. FOC: $100 = (1+y)^2 \Rightarrow y^* = 9 \Rightarrow x^* = 9/n$.

(b) Strictly concave welfare function. FOC: $100n = (1+y)^2 \Rightarrow \hat{y} = 10\sqrt{n} - 1 \Rightarrow \hat{x} = (10\sqrt{n} - 1)/n$. Thus $\hat{x}/x^* = (10\sqrt{n} - 1)/9$, increasing in $n \geq 1$ from 1 at $n = 1$.

3. Consider the following extensive form game:



(a) Find the corresponding strategic (i.e., normal form) game. **[1 pt]**

(b) Find all subgame perfect equilibria in behavioral strategies. **[2 pts]**

(c) Find all sequential equilibria. **[3 pts]**

Solution (a)

	c	d
(a, e)	$1, 0$	$1, 0$
(a, f)	$1, 0$	$1, 0$
(b, e)	$1, 0$	$0, 1$
(b, f)	$0, 3$	$1, 1$

(b)
- There are two subgames: the game as a whole and a proper subgame starting at the decision node of player 2.

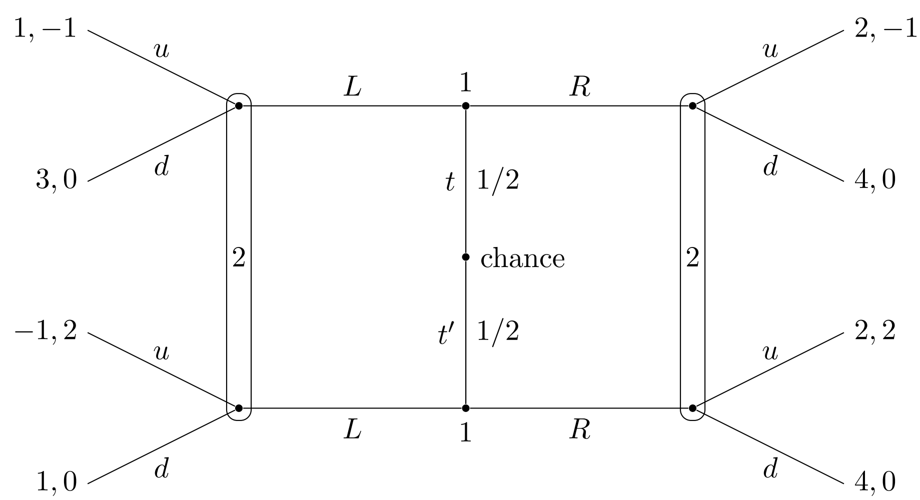
- The latter game

	c	d
e	$1, 0$	$0, 1$
f	$0, 3$	$1, 1$

  has a unique Nash equilibrium where e is chosen with probability $2/3$ and c is chosen with probability $1/2$. Player 1's expected payoff in this equilibrium is $1/2$. So in the game as a whole, it is optimal for player 1 to choose a with probability 1.
- Conclusion: There is a unique subgame perfect equilibrium in behavioral strategies where player 1 chooses a with probability 1 and e with probability $2/3$ and player 2 chooses c with probability $1/2$.

(c)
- Strategies: The behavioral strategies in a sequential equilibrium must be subgame perfect, so (b) gives only one candidate.
- Belief system: Any completely mixed profile of behavioral strategies in which the probability of c approaches $1/2$ must assign equal probability to both nodes in player 1's information set.
- Conclusion: there is a unique candidate for a sequential equilibrium: the profile of behavioral strategies in (b) with a belief system assigning probability $1/2$ to both nodes in 1's information set. Given the existence theorem of sequential equilibria, this is the game's unique sequential equilibrium.

4. Consider the signaling game below:



(a) Find, if any, the separating equilibria where 1 chooses R if chance selects t, but L if chance selects t'. **[2 pts]**

(b) Find, if any, the pooling equilibria where 1 chooses R both if chance selects t and if chance selects t'. **[2 pts]**

Solution (a)
- Denoting 1's strategy as (R, L), player 2's best reply is to choose u in the left information set and d in the right, denoted (u, d).
- But 1's strategy (R, L) is not a best reply to (u, d): it is better to choose R if chance selects t'.
- Conclusion: no such equilibria.

(b)
- If player 1 plays (R, R), 2 assigns equal probability to both nodes in the right information set, making u the unique best reply there. So there are two cases:

- Is $((R,R),(u,u))$ a pooling equilibrium? The strategy of pl. 1 is sequentially rational and so is 2's strategy in the right information set. Bayesian consistency imposes no restrictions on beliefs in the left information set, but sequential rationality of action u requires that the probability assigned to the top node there lies in $[0,2/3]$.
- Is $((R,R),(d,u))$ a pooling equilibrium? No! Pl. 1's best response to (d,u) is to choose L after chance selects t.
- Conclusion: pooling equilibria $((R,R),(u,u))$ where 2 assigns probability $\alpha \in [0,2/3]$ to the top node in the left information set and probability $1/2$ to the nodes in the right information set.

Part B – Combinatorial Game Theory

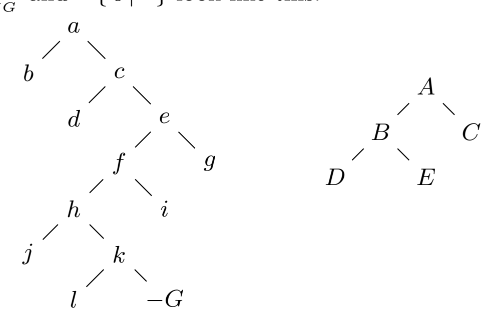
Jonas Sjöstrand

5. For any game G, we define the game $+_G := \{\, 0 \;\|\; 0 \,|\, -G \,\}$.

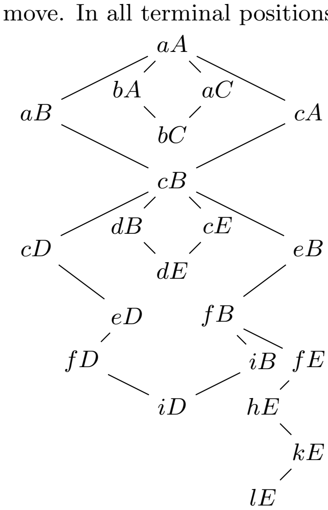
(a) Show that $+_{+_{+_G}} = \{\, 0 \,|\, * \,\}$ for any game G. **[2 pts]**

(b) Find all games G such that $+_G = G$. **[1 pt]**

Solution (a) The game trees for $+_{+_{+_G}}$ and $-\{\, 0 \,|\, * \,\}$ look like this:



A position in the sum $H = +_{+_{+_G}} - \{\, 0 \,|\, * \,\}$ will be denoted by a two-letter combination xX where x is a node in the game tree of $+_{+_{+_G}}$ and X is a node in the game tree of $-\{\, 0 \,|\, * \,\}$. The game starts at aA. The following diagram shows that the second player has a winning strategy. For the first player every option is shown, and for the second player only one option from each position is shown — a winning move. In all terminal positions, the first player is at turn.



(b) If $+_G = G$ then $+_{+_{+_G}} = +_{+_G} = +_G = G$ so, by (a), G must be equal to $\{\,0\,|\,*\,\}$. To see that $+_{\{\,0\,|\,*\,\}} = \{\,0\,|\,*\,\}$ we draw the game trees of $+_{\{\,0\,|\,*\,\}}$ and $-\{\,0\,|\,*\,\}$ and do the same reasoning as in (a). In fact, we only have to omit the nodes j, k, l and $-G$ from the tree in (a) and the nodes kE and lE from the strategy diagram!

6. The game *Toppling dominoes* is played with a row of black and white dominoes. In each move, the player chooses a domino and topples it either left or right. Every domino in that direction also topples and is removed from the game. Left may only choose a black domino and Right may only choose a white one. Here is an example of a game of Toppling dominoes:

$$\blacksquare\square\blacksquare\blacksquare\blacksquare\square \xrightarrow{L} \blacksquare\square\blacksquare\blacksquare \xrightarrow{R} \blacksquare \xrightarrow{L} \varnothing$$

Prove that the following equalities hold for any nonnegative integers m and n.

$$\underbrace{\blacksquare\cdots\blacksquare}_{m+1}\underbrace{\square\cdots\square}_{n+1} = \{\,m\,|\,-n\,\}$$

$$\underbrace{\blacksquare\square\blacksquare\cdots\square}_{2n} = *n$$

$$\underbrace{\blacksquare\square\blacksquare\cdots\square\blacksquare}_{2n+1} = \frac{1}{2^n}$$

$$\blacksquare\underbrace{\square\square\cdots\square}_{n+2}\blacksquare = \{\,0\ \|\ 0\,|\,-n\,\}$$

[4 pts]

Solution Let us introduce the following notation.

$$A_{m,n} := \underbrace{\blacksquare\cdots\blacksquare}_{m+1}\underbrace{\square\cdots\square}_{n+1}$$

$$B_n := \underbrace{\blacksquare\square\blacksquare\cdots\square}_{2n}$$

$$C_n := \underbrace{\blacksquare\square\blacksquare\cdots\square\blacksquare}_{2n+1}$$

$$D_n := \blacksquare\underbrace{\square\square\cdots\square}_{n+2}\blacksquare$$

We will prove all equalities by induction. They clearly hold for $m = n = 0$.

$$A_{m,n} = \{\,A_{m-1,n},\ldots,A_{0,n},-(n+1),m,m-1,\ldots,0\ |\ A_{m,n-1},\ldots,A_{m,0},m+1,-n,-(n-1),\ldots,0\,\}$$

which by induction is equal to

$$\{\,\{\,m-1\,|\,-n\,\},\ldots,\{\,0\,|\,-n\,\},-(n+1),m,m-1,\ldots,0\,|\,\{\,m\,|\,-(n-1)\,\},\ldots,\{\,m\,|\,0\,\},m+1,-n,-(n-1),\ldots,0\,\}.$$

Deleting all dominated options yields $A_{m,n} = \{\,m\,|\,-n\,\}$.

Similarly,

$$\begin{aligned} B_n &= \{\,-C_{n-1},\ldots,-C_0,B_{n-1},\ldots,B_0\ |\ C_{n-1},\ldots,C_0,B_{n-1},\ldots,B_0\,\} \\ &\overset{\text{ind}}{=} \{\,-\frac{1}{2^{n-1}},\ldots,-\frac{1}{2^0},*(n-1),\ldots,*0\ |\ \frac{1}{2^{n-1}},\ldots,\frac{1}{2^0},*(n-1),\ldots,*0\,\} \\ &= \{\,*(n-1),\ldots,*0\ |\ *(n-1),\ldots,*0\,\} \\ &= *n \end{aligned}$$

and

$$\begin{aligned}
C_n &= \{\, B_n, \ldots, B_0 \,|\, C_{n-1}, \ldots, C_0 \,\} \\
&\overset{\text{ind}}{=} \{\, *n, \ldots, *0 \,|\, \frac{1}{2^{n-1}}, \ldots, \frac{1}{2^0} \,\} \\
&= \{\, *n, \ldots, *0 \,|\, \frac{1}{2^{n-1}} \,\} \\
&= \frac{1}{2^n},
\end{aligned}$$

where the last equality follows from the simplicity theorem.

Finally,

$$\begin{aligned}
D_n &= \{\, A_{0,n+1}, 0 \,|\, A_{0,n}, \ldots, A_{0,0}, 1 \,\} \\
&\overset{\text{ind}}{=} \{\, \{\, 0 \,|\, -(n+1) \,\}, 0 \mid \{\, 0 \,|\, -n \,\}, \ldots, \{\, 0 \,|\, 0 \,\} \,\} \\
&= \{\, 0 \mid \{\, 0 \,|\, -n \,\} \,\},
\end{aligned}$$

where the last equality follows from deleting dominated options.

7. Let $G = \{\, 5 \,|\, 3 \;\|\; 0 \,\}$ and $H = \{\, 0 \,|\, -4 \,\}$. What are the temperatures of G, H and $G + H$? **[4 pts]**

Solution We have $G_t = \{\{5|3\}_t - t \,|\, t\}$ unless this expression is a number. Clearly,

$$\{5|3\}_t = \begin{cases} \{5 - t \,|\, 3 + t\} & \text{if } 0 \le t \le 1, \\ 4 & \text{if } t > 1, \end{cases}$$

so, by the translation theorem,

$$\{\{5|3\}_t - t \,|\, t\} = \begin{cases} \{5 - 2t \,|\, 3 \;\|\; t\} & \text{if } 0 \le t \le 1, \\ \{4 - t \,|\, t\} & \text{if } t > 1, \end{cases}$$

and

$$G_t = \begin{cases} \{5 - 2t \,|\, 3 \;\|\; t\} & \text{if } 0 \le t \le 1, \\ \{4 - t \,|\, t\} & \text{if } 1 < t \le 2, \\ 2 & \text{if } t > 2. \end{cases}$$

We conclude that the temperature of G is 2.

The game H is even easier to cool:

$$H_t = \begin{cases} \{-t \,|\, t - 4\} & \text{if } 0 \le t \le 2, \\ -2 & \text{if } t > 2, \end{cases}$$

and its temperature is 2.

Cooling is linear, so

$$(G + H)_t = G_t + H_t = \begin{cases} \{5 - 2t \,|\, 3 \;\|\; t\} + \{-t \,|\, t - 4\} & \text{if } 0 \le t \le 1, \\ 0 & \text{if } t > 1. \end{cases}$$

It is easy to check that $(G + H)_1 = \{3|3 \;\|\; 1\} - \{3|1\} \neq 0$, so the temperature of $G + H$ is 1.

8. The impartial game Kayles is played with a row of pins, possible containing some gaps. A move consists in throwing a ball that removes either one or two adjacent pins. Example:

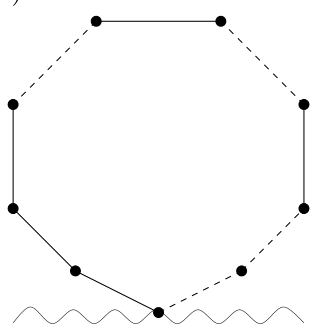
$$\underline{\dagger\,\dagger\,\dagger\,\dagger\,\dagger} \to \underline{\dagger\quad\;\dagger\,\dagger} \to \underline{\dagger\qquad\quad} \to \underline{\qquad\qquad}$$

Compute the Grundy value of K_5, a row of 5 pins: $\underline{\dagger\,\dagger\,\dagger\,\dagger\,\dagger}$. **[2 pts]**

Solution We let K_n denote a row of n adjacent pins and compute the Grundy values of $K_0, K_1, \ldots, K_5$ by using the mex rule and nim addition.

$$K_0 = *0,$$
$$K_1 = \{K_0\} = \{*0\} = *1,$$
$$K_2 = \{K_0, K_1\} = \{*0, *1\} = *2,$$
$$K_3 = \{K_1, K_2, K_1 + K_1\} = \{*1, *2, *1 + *1\} = \{*1, *2, 0\} = *3,$$
$$K_4 = \{K_2, K_1 + K_1, K_3, K_2 + K_1\} = \{*2, 0, *3, *2 + *1\} = \{*2, 0, *3, *3\} = *1,$$
$$K_5 = \{K_3, K_2 + K_1, K_4, K_3 + K_1, K_2 + K_2\} = \{*3, *3, *1, *2, *0\} = *4.$$

9. Compute the value of the following Blue-Red Hackenbush position. (Solid edges are blue and dashed edges are red.) **[2 pts]**



Solution The sign-expansion of a Hackenbush path is obtained by walking along the path, starting at the ground, and writing a plus for each blue edge and a minus for each red edge. Thus, Left has the following options from the depicted position:

$$(--+-+-++) = -2 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} + \frac{1}{64},$$
$$(+) + (--+-+-+) = 1 + (-2 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32}),$$
$$(++) + (--+-+-) = 2 + (-2 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16}),$$
$$(+++-) + (--+-) = (3 - \frac{1}{2}) + (-2 + \frac{1}{2} - \frac{1}{4}),$$
$$(+++-+-) + (--) = (3 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8}) - 2.$$

The largest of these is $(3 - \frac{1}{2}) + (-2 + \frac{1}{2} - \frac{1}{4}) = \frac{3}{4}$.

Right's options are

$$(+++) + (--+-+) = 3 + (-2 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8}),$$
$$(+++-+) + (--+) = (3 - \frac{1}{2} + \frac{1}{4}) + (-2 + \frac{1}{2}),$$
$$(+++-+-+) + (-) = (3 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16}) - 1,$$
$$(+++-+-+-) = 3 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32},$$

of which $(3 - \frac{1}{2} + \frac{1}{4}) + (-2 + \frac{1}{2}) = \frac{5}{4}$ is the smallest one.

We conclude that the value of the position is $\{\frac{3}{4} \mid \frac{5}{4}\} = 1$.