SF2972 Game Theory
Exam with Solutions
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PART A – CLASSICAL GAME THEORY

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1. Consider the two-player normal-form game

	a_2	b_2	c_2
a_1	$6,2$	$1,0$	$0,3$
b_1	$0,2$	$4,3$	$0,0$
c_1	$7,0$	$4,0$	$1,1$

(a) Find all pure strategies that are weakly dominated (by a pure or mixed strategy). **[1 pt]**
(b) Find all pure strategies that are strictly dominated (by a pure or mixed strategy). **[1 pt]**
(c) Find all rationalizable pure strategies. **[1 pt]**
(d) Find all pure-strategy Nash equilibria. **[1 pt]**
(e) Find all pure-strategy perfect equilibria. **[1 pt]**

Solution
(a) a_1 and b_1
(b) a_1
(c) b_1, c_1, b_2 and c_2
(d) (b_1, b_2) and (c_1, c_2)
(e) (c_1, c_2)

2. Consider two ice-cream vendors, A and B, who sell the same ice-cream (say, Magnum Classic) to consumers who are uniformly distributed on a 1 kilometers long beach. Let $X = [0, 1000]$ represent the beach. The vendors have no fixed costs but each vendor has a unit cost of c euros per ice-cream and sells each ice-cream at a fixed and given price $p > c$. Each vendor has to choose a location, x_A and x_B, respectively, in X. Each consumer buys exactly one ice-cream from the seller nearest to his or her location (and with equal probability from either A or B in case they happen to be at the same distance from the consumer).

(a) Draw a picture of the set X, indicate two (arbitrary) distinct locations, $x_A < x_B$, for the ice-cream vendors, and indicate in the diagram, and define also algebraically, which consumers will buy from which ice-cream vendor. **[1 pt]**

(b) For any given locations, $x_A < x_B$, define each vendor's profit (units sold times net revenue on each ice-cream) as a function of x_A and x_B. **[1 pt]**

(c) Suppose you could choose the locations for the ice-cream vendors, and your goal was to minimize the average distance for consumers to their nearest ice-cream vendor. What locations would you then choose? **[1 pt]**

(d) Suppose instead the two vendors are free to choose their locations in X, and that they would do so simultaneously, that is independently of each other, without knowing what location the other vendor chooses. (They may happen to choose the same location, in which case they will sell equally many units.) Define this as a normal-form game, that is, the players, their (pure) strategy sets, and their payoff functions. Show that these payoff functions are discontinuous, that the players do not always have a best reply to each other's strategy, but that there nevertheless exists a unique Nash equilibrium. Find this! Do the vendors earn higher or lower profits than under your proposal in (c)? **[2 pts]**

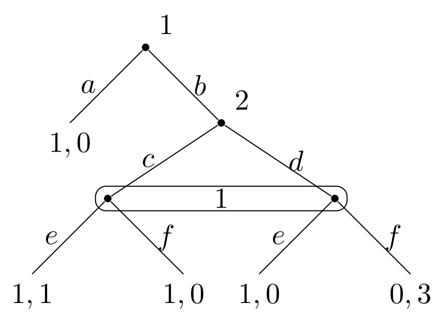
Solution (a) Consumers located to the left of $(x_A + x_B)/2$ will buy from A and the others from B.

(b) $\pi_A(x_A, x_B) = D \cdot (p - c)(x_A + x_B)/2$ and $\pi_A(x_A, x_B) = D \cdot (p - c)(1000 - (x_A + x_B)/2)$, where $D > 0$ is the population size.

(c) Minimization of total travel distance to nearest vendor gives $x_A = 250$ and $x_B = 750$.

(d) Payoff functions are discontinuous at $x_A = x_B \neq 500$. For $x_B \neq 500$, A has no best reply ("wants" to be very close to B on the side nearest the midpoint). At $x_A = x_B = 500$ we obtain the unique Nash equilibrium.

3. Consider the following extensive form game:



(a) Find the corresponding strategic (i.e., normal form) game. **[1 pt]**

(b) Find all subgame perfect equilibria in behavioral strategies. **[2 pts]**

(c) Find all sequential equilibria. **[3 pts]**

Solution (a)

	c	d
(a, e)	$1, 0$	$1, 0$
(a, f)	$1, 0$	$1, 0$
(b, e)	$1, 1$	$1, 0$
(b, f)	$1, 0$	$0, 3$

(b) – There are two subgames: the game as a whole and a proper subgame starting at the decision node of player 2.

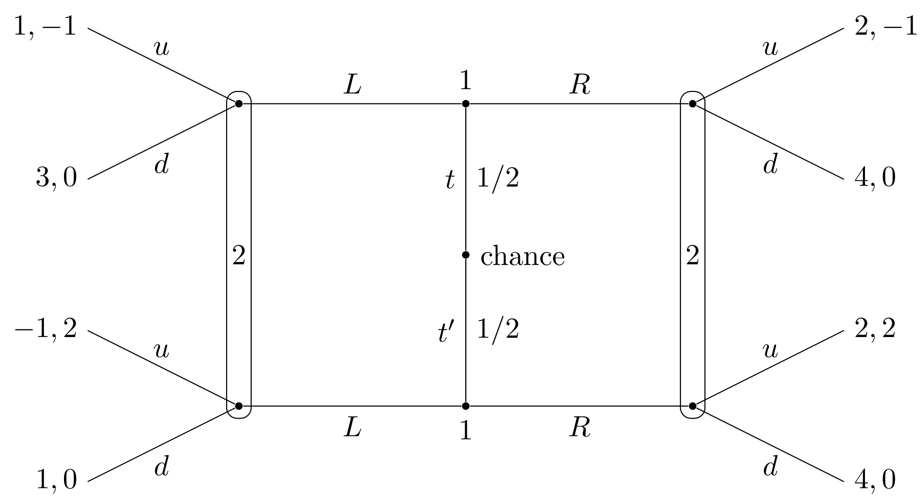
- The latter game

	c	d
e	$1,1$	$1,0$
f	$1,0$	$0,3$

  has infinitely many Nash equilibria where e is chosen with probability $p \in [3/4, 1]$ and c is chosen with probability 1. Player 1's expected payoff in this equilibrium is 1. That is the same payoff player 1 gets from choosing a, so in the initial node any probability assigned to a will give a best reply.
- Conclusion: There are infinitely many subgame perfect equilibria in behavioral strategies where player 1 chooses a with $p_a \in [0, 1]$ and e with probability $p \in [3/4, 1]$ and player 2 chooses c with probability 1.

(c)
- Strategies: The behavioral strategies in a sequential equilibrium must be subgame perfect, so (b) gives the candidates.
- Belief system: Any completely mixed profile of behavioral strategies in which the probability of c is p_c must assign probability p_c to the left node in player 1's information set. A standard limit argument then shows that the strategies in (b) with the belief system just specified.

4. Consider the signaling game below:



(a) Find, if any, the separating equilibria where 1 chooses L if chance selects t, but R if chance selects t'. **[2 pts]**

(b) Find, if any, the pooling equilibria where 1 chooses L both if chance selects t and if chance selects t'. **[2 pts]**

Solution (a)
- Denoting 1's strategy as (L, R), player 2's best reply is to choose d in the left information set and u in the right, denoted (d, u).
- It is easy to check that also (L, R) is a best reply to (d, u), so the pair $((L, R), (d, u))$ is a candidate for a separating equilibria.
- A Bayesian consistent belief system of player 2 must then assign probability 1 to being in the top (bottom) node of the left (right) information set.
- Conclusion: $((L, R), (d, u))$ with the belief system above is a separating equilibrium.

(b)
- If player 1 plays (L, L), player 2 assigns equal probability to both nodes in the left information set, making u the unique best reply there.
- So player 1's expected payoff in the left information set is zero.
- But then deviating to (R, R), regardless of what player 2 does in the right information set, gives a strictly higher payoff.

- Conclusion: no such equilibria.

The pure equilibria also follow from the associated strategic form game, but the reasoning above was probably simpler.

	(u,u)	(u,d)	(d,u)	(d,d)
(L,L)	$0, \frac{1}{2}^*$	$0, \frac{1}{2}^*$	$2, 0$	$2, 0$
(L,R)	$\frac{3}{2}, \frac{1}{2}$	$\frac{5}{2}, -\frac{1}{2}$	$\frac{5}{2}^*, 1^*$	$\frac{7}{2}, 0$
(R,L)	$\frac{1}{2}, \frac{1}{2}$	$\frac{3}{2}, 1^*$	$\frac{3}{2}, -\frac{1}{2}$	$\frac{5}{2}, 0$
(R,R)	$2^*, \frac{1}{2}^*$	$4^*, 0$	$2, \frac{1}{2}^*$	$4^*, 0$

Part B – Combinatorial Game Theory

Jonas Sjöstrand

5. A game G is said to be *small* if $-x < G < x$ for any positive number x, and it is said to be *all small* if all its positions are small.

Let $G = \{\ 0\ \|\ 0\,|\,{-1}\ \}$.

(a) Show that G is positive. **[1 pt]**

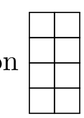
(b) Show that G is small. **[1 pt]**

(c) Show that G is smaller than any positive all small game. **[2 pts]**

Solution (a) If Left starts she wins by moving to 0. If Right starts Left wins by moving to 0. Since Left wins whoever starts, the game is positive.

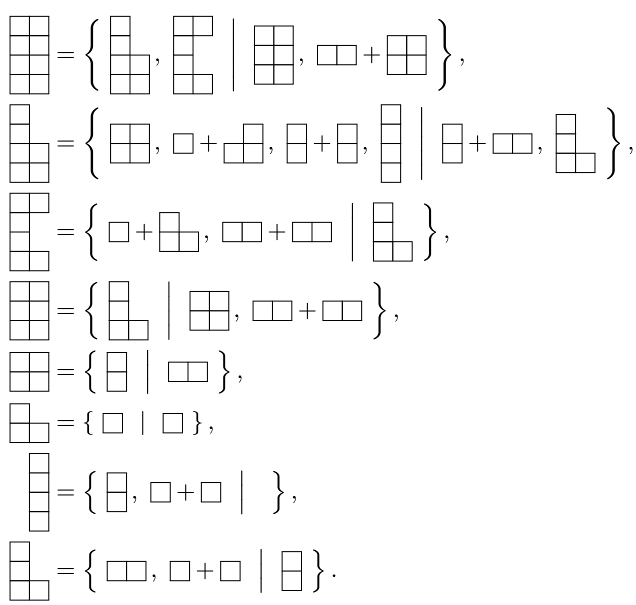
(b) For any positive number x we must show that the game $x - G = x + \{\ 1\,|\,0\ \|\ 0\ \}$ is positive, that is, that Left will win whoever starts. By the weak number avoidance theorem we may assume that the players move in a non-number component if there is any. Thus, if Right starts he moves to $x + 0$, and if Left starts she moves to $x + \{1|0\}$ and then Right will move to $x + 0$. Since x is positive Left will win the game.

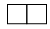
(c) Let x be any positive all small game and consider $x - G = x + \{\ 1\,|\,0\ \|\ 0\ \}$. Left's move to $x + \{1|0\}$ is good since if Right moves to $x + 0$ Left will win (since x is positive) and if Right moves to some $x^R + \{1|0\}$ Left will move to $x^R + 1$ which is positive since $-1 < x^R$ by the all small assumption. If Right starts from $x - G$, he may move to $x + 0$ in which case Left will win, or he may move to some $x^R + \{\ 1\,|\,0\ \|\ 0\ \}$ wherafter Left moves to $x^R + \{1|0\}$. Then, if Right moves to $x^R + 0$ Left will win since $x^R \triangleright 0$, and if Right moves to some $x^{RR} + \{1|0\}$ Left will move to $x^{RR} + 1$ which is positive by the all small assumption.

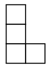
6. Compute the value of the Domineering position  and write it in canonical form.

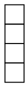
[4 pts]

Solution We have

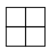


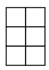
Plugging  $= 0$,  $= 1$ and  $= -1$ into the above equations yields, starting from below,

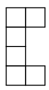
 $= \{ -1 , 0 \mid 1 \} = \dfrac{1}{2}$,

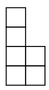
 $= \{ 1 , 0 \mid \ \} = 2$,

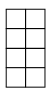
 $= \{ 0 \mid 0 \} = *$,

 $= \{ 1 \mid -1 \}$,

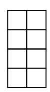
 $= \left\{ \dfrac{1}{2} \;\middle|\; \{1|-1\}, -2 \right\} = \{\tfrac{1}{2}|-2\}$ (since $\{1|-1\}$ is dominated),

 $= \left\{ *, -2 \;\middle|\; \dfrac{1}{2} \right\} = 0$ (by the simplicity theorem),

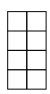
 $= \left\{ \{1|-1\}, *, 2, 2 \;\middle|\; 0, \dfrac{1}{2} \right\} = \{2|0\}$ (since $\{1|-1\}$, $*$ and $\dfrac{1}{2}$ are dominated),

 $= \left\{ \{2|0\}, 0 \;\middle|\; \{\tfrac{1}{2}|-2\}, -1 + \{1|-1\} \right\}$.

By the translation theorem, the right option $-1 + \{1|-1\}$ equals $\{0|-2\}$ which dominates the other right option $\{\frac{1}{2}|-2\}$, so

 $= \{ \{2|0\}, 0 \mid \{0|-2\} \}$.

Since Left will win the game if Right starts, the game is greater than or equal to zero and the left option $\{2|0\}$ is reversible through 0. After replacing $\{2|0\}$ by all left options of 0, that is, by nothing, we finally obtain

 $= \{ 0 \mid \{0|-2\} \}$.

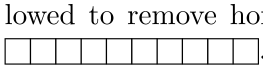
which is in canonical form.

7. (a) Show that $t(G) > t(H) \Rightarrow t(G + H) = t(G)$. **[2 pts]**

(b) Find games G and H such that $t(G + H) < \max\{t(G), t(H)\}$. (You don't have to find *all* such games — one example is enough.) **[1 pt]**

Solution (a) Suppose $t(G) > t(H)$. Then, by definition of temperature, $H_{t(G)}$ is a number but $G_{t(G)}$ is not. This implies that the sum $G_{t(G)} + H_{t(G)}$ is not a number. (If it were a number x, then $G_{t(G)} = x - H_{t(G)}$ would be a sum of two numbers and hence a number.) By linearity of cooling, $(G + H)_{t(G)} = G_{t(G)} + H_{t(G)}$, so $(G + H)_{t(G)}$ is not a number and hence $t(G) \le t(G + H)$. On the other hand we know that $t(G + H) \le \max\{t(G), t(H)\} = t(G)$ so we conclude that $t(G + H) = t(G)$.

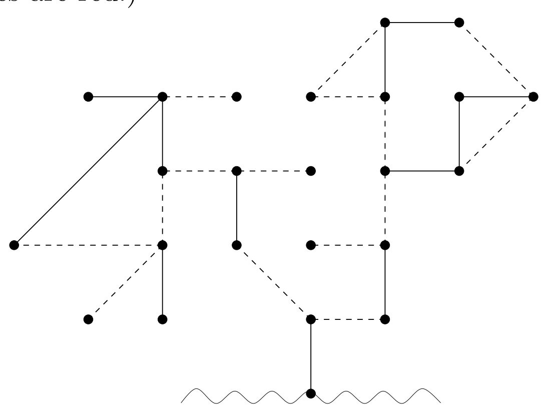
(b) We can choose $G = H = \{1|-1\}$. They have temperature 1 but their sum is zero.

8. Consider the impartial variant of the game Domineering where both players are allowed to remove horizontal dominoes. Compute the Grundy value of the position . **[2 pts]**

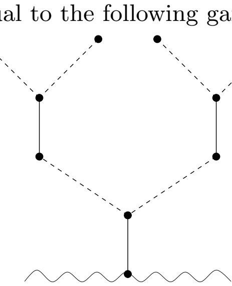
Solution We let D_n denote a row of n boxes and compute the Grundy values of $D_1, D_2 \ldots, D_{10}$ by using the mex rule and nim addition.

$$
\begin{aligned}
D_1 &= 0,\\
D_2 &= *,\\
D_3 &= *,\\
D_4 &= \{D_2, D_1 + D_1\} = \{*, 0+0\} = *2,\\
D_5 &= \{D_3, D_2 + D_1\} = \{*, *+0\} = 0,\\
D_6 &= \{D_4, D_3 + D_1, D_2 + D_2\} = \{*2, *+0, *+*\} = \{*2, *, 0\} = *3,\\
D_7 &= \{D_5, D_4 + D_1, D_3 + D_2\} = \{0, *2+0, *+*\} = \{*2, 0\} = *,\\
D_8 &= \{D_6, D_5 + D_1, D_4 + D_2, D_3 + D_3\} = \{*3, 0+0, *2+*, *+*\} = \{*3, 0\} = *,\\
D_9 &= \{D_7, D_6 + D_1, D_5 + D_2, D_4 + D_3\} = \{*, *3+0, 0+*, *2+*\} = \{*3, *\} = 0,\\
D_{10} &= \{D_8, D_7 + D_1, D_6 + D_2, D_5 + D_3, D_4 + D_4\} = \{*, *+0, *3+*, 0+*, *2+*2\} = \{*2, *, 0\} = *3.
\end{aligned}
$$

9. Compute the value of the following Blue-Red Hackenbush position. (Solid edges are blue and dashed edges are red.) **[2 pts]**



Solution For symmetry reasons the game is equal to the following game.



The game value is $G = 1 : (H + H)$ where $H = -1 : (1 : -2) = -1 : \frac{1}{4} = -\frac{7}{8}$, so $G = 1 : -\frac{7}{4} = \frac{5}{16}$.