SF2972 Game Theory
Exam
June 3, 2013

Time: 14.00-19.00
No permitted aids
Examiner: Boualem Djehiche

The exam is divided in two: Part A on classical game theory and Part B on combinatorial game theory. The maximal score is 20 points from part A and 15 points from part B, and your grade is calculated as follows:

Minimal score:	17	18	21	24	27	30
Grade:	Fx	E	D	C	B	A

Write clearly and concisely, give precise definitions of game-theoretic concepts and precise statements of game-theoretic results that you refer to. Provide precise derivations and motivations for your answers.

Part A – Classical Game Theory

Jörgen Weibull and Mark Voorneveld

1. Consider the two-player game G with normal form

	L	M	R
A	$8, 11$	$-3, 0$	$0, 0$
B	$9, -1$	$4, 1$	$0, 0$
C	$0, -2$	$0, 0$	$1, 4$

(a) For an arbitrary finite game in normal form: Give exact definitions of *weak* and *strict dominance* (for mixed strategies), *rationalizability* (of pure strategies), and *Nash equilibrium* (in mixed strategies).
(b) Find all pure strategies that are *strictly dominated* by a pure or mixed strategy.
(c) Find all *rationalizable* pure strategies.
(d) Find all *Nash equilibria*, in pure and mixed strategies.
(e) Define *perfect equilibrium* and find all such equilibria, in pure and mixed strategies.

[6 pts]

2. A group with $n \geq 1$ members together own a production unit. Each member i chooses an input level $x_i \geq 0$. Total output depends on total input, $x_1 + \cdots + x_n$. Output is a public good enjoyed by all group members and each member's input is costly to the member. The resulting utility level for each member i is

$$u_i(x_1, \ldots, x_n) = 2\sqrt{x_1 + \cdots + x_n} - x_i$$

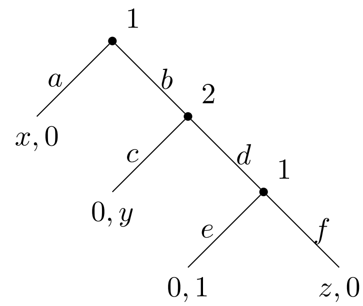
Each member has to choose his or her input without observing the others' inputs.

(a) Show that the game has infinitely many Nash equilibria in pure strategies. (A precise and rigorous demonstration is required.) Solve for the aggregate input level (the sum of inidividual inputs in equilibrium). Is it increasing or decreasing in n, or independent of n? Explain your findings! **[2 pts]**
(b) Solve task (a) for two *altruistic* group members, that is, a group with two members who care (positively) about each others' utility. More exactly, let $n = 2$ and assume that group member i's (total) utility (with a capital "U") is given by

$$\begin{aligned} U_i(x_1, x_2) &= u_i(x_1, x_2) + \alpha_i \cdot u_j(x_1, x_2) \\ &= 2(1 + \alpha_i)\sqrt{x_1 + x_2} - x_i - \alpha_i x_j \end{aligned}$$

for $0 < \alpha_1 \leq \alpha_2 \leq 1$, the members' degrees of altruism, and for $i = 1, 2$ and $j \neq i$. **[2 pts]**

3. Consider the game below:



Find all values of $x, y, z \in \mathbb{R}$ such that:

(a) $((a, e), c)$ is a Nash equilibrium. **[1 pt]**

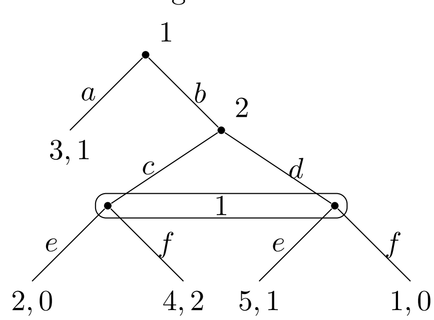
(b) $((a, e), c)$ is a subgame perfect equilibrium. **[1 pt]**

4. Use the deferred acceptance algorithm to find a stable matching in the marriage problem with ranking matrix:

	w_1	w_2	w_3	w_4
m_1	$1,4$	$2,4$	$3,4$	$4,4$
m_2	$1,3$	$2,3$	$3,3$	$4,3$
m_3	$1,2$	$2,2$	$3,2$	$4,2$
m_4	$1,1$	$2,1$	$3,1$	$4,1$

[1 pt]

5. Consider the following extensive form game:



(a) Find the corresponding strategic (i.e., normal form) game. **[1 pt]**

(b) Find all pure-strategy Nash equilibria. **[1 pt]**

(c) Find all subgame perfect equilibria in behavioral strategies. **[3 pts]**

(d) Find all sequential equilibria. **[2 pts]**

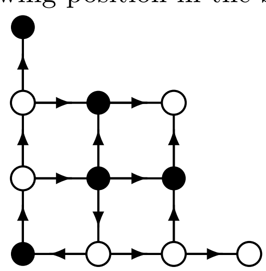
PART B – COMBINATORIAL GAME THEORY

Jonas Sjöstrand

6. In a directed graph, a vertex with no outgoing edges is called a *sink*.

The *Sink Removal Game* is a partizan game played on a directed graph where each vertex is colored either black or white. The players alternate moves, and in each move the player chooses a sink and removes it (and all its ingoing edges). Left can only remove white sinks and Right can only remove black sinks. If no legal move is available, the player at turn will lose the game.

(a) Find the value of the following position in the Sink Removal Game. **[2 pts]**



(b) Show that the value of any position G in the Sink Removal Game is a number. Hint: Show that $G^L < G < G^R$ for any left option G^L and any right option G^R. **[3 pts]**

7. Find a game G such that

- G has temperature 5,
- G has mean value 1,
- G is fuzzy to 3,
- G is fuzzy to -1, and
- $-3 < G < 5$.

Note that there might exist several G with these properties. You are only acquired to find one such game. **[4 pts]**

8. Consider the game of Nim with the additional rule that we are only allowed to remove one or four sticks in each move.

(a) Find the Grundy value $g(P_n)$ of a pile P_n of n sticks, for $0 \leq n \leq 9$. **[1 pt]**

(b) Find $g(P_n)$ for general n. **[1 pt]**

(c) Find a winning move from the three-pile position $(100, 49, 18)$. **[1 pt]**

9. (a) Construct a Blue-Red Hackenbush position that has value 18/32. **[1 pt]**

(b) A *full binary tree* is an acyclic graph where each vertex has degree one or three. (The *degree* of a vertex is the number of edges incident to it. *Acyclic* means that the graph has no cycles.) Construct a Blue-Red Hackenbush position with value 5/8 that is a full binary tree with one leaf (a vertex of degree one) attached to the ground. **[2 pts]**