KTH Mathematics

Examination in SF2974 Portfolio Theory and Risk Management, January 7, 2009, 8:00–13:00.

Examiner: Filip Lindskog, tel. 790 7217, e-mail: lindskog@math.kth.se

Allowed technical aids: calculator.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

Good luck!

- - - - - - - - - - - - - - - - - - - -

Problem 1

Consider a market with the three risk-free bonds shown in Table 1. Show that it is impossible to find a positive forward rate for the time period one year from now until 18 months from now that makes the market arbitrage free and complete **or** show the opposite by determining such a forward rate. (10 p)

Bond	A	B	C
Bond price ($)	94.9	96	176.1
Maturity (years)	0.5	1.5	2
Annual coupon ($)	0	8	10
Face value ($)	100	100	200

Table 1: Bond specifications. Half of the annual coupon is paid every six months from today and including the time of maturity. The first coupon payment is in six months.

Problem 2

Consider a market with risky assets and a corresponding portfolio frontier shown in Figure 1. Suppose further that a risk-free asset with a non-negative rate of return is introduced on this market so that the Sharpe ratio of the portfolios on the efficient frontier is maximized. Determine the Sharpe ratio and the rate of return for the risk-free asset. (10 p)

Problem 3

Consider an economy satisfying the CAPM exactly with two stocks specified, for the next one-year period, in Table 2. The linear correlation coefficient between the returns of stocks A and B is 50%. Determine the risk-free rate of return per year in this economy. (10 p)

	Num. of shares	Share price	Exp. return	St. dev. of return
Stock A	1000	$15	5%	15%
Stock B	2000	$5	10%	30%

Table 2: Number of outstanding shares, share price, expected return and standard deviation of return for Stock A and B, respectively.

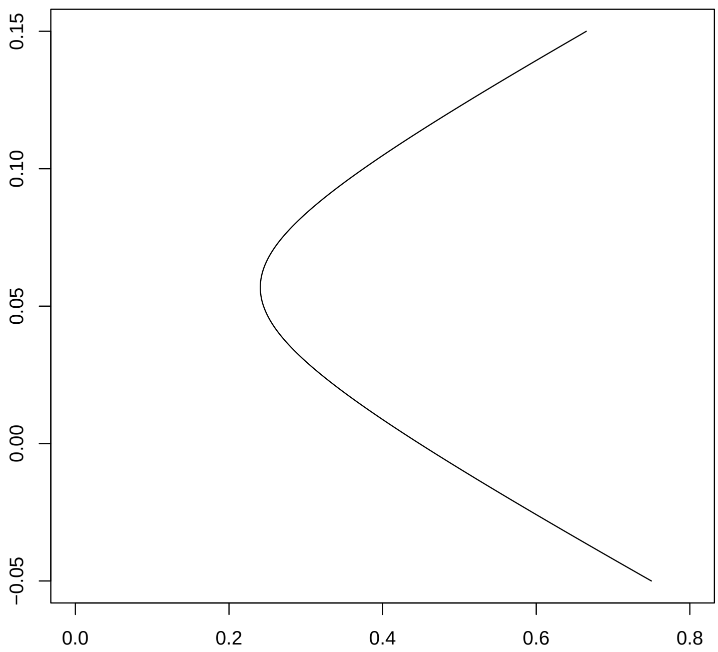

Figure 1: Portfolio frontier.

Problem 4

Consider a one-period model for a market with two assets A and B, whose current spot prices are both \$100. For each asset it is assumed that one year from now the price has moved to either \$150 or \$50. The probability of an up-up-move and the probability of a down-down-move is assumed to be 30%. The probability of an up-down-move is assumed to be 20%. Consider an investor with an initial wealth of \$100 which she wants to invest fully by taking positions in the two assets (positions corresponding to fractions of the assets are allowed). The investor's portfolio will be held for one year and shorting is allowed and unrestricted. The investor's utility function is given by $U(x) = -e^{-x/100}$ and the investor wants to invest her capital so that the expected utility of her wealth one year from now is maximized.

(a) Formulate the investor's optimization problem with mathematics as precisely as possible. (5 p)

(b) Solve it. (5 p)

Problem 5

Consider loss variables L with finite variances and the risk measure ρ given by $\rho(L) = \mathrm{E}[L] + 2\sqrt{\mathrm{Var}(L)}$. A coherent risk measure satisfies the properties (T) translation invariance, (PH) positive homogeneity, (M) monotonicity and (S) subadditivity. For each of these properties, show that ρ satisfies the property or give a counterexample showing that it does not. (10 p)

Problem 1

The bond prices may be expressed in terms of the discount factors as

$$\begin{pmatrix} 100 & 0 & 0 & 0 \\ 4 & 4 & 104 & 0 \\ 5 & 5 & 5 & 205 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} 94.9 \\ 96 \\ 176.1 \end{pmatrix}$$

This gives $d_1 = 0.949$ and hence $d_2 = 24 - d_1 - 26d_3 = 23.051 - 26d_3$. Substituting this into the last equation gives $41d_4 - 25d_3 = 176.1/5 - 24$. Hence, $d_4 \approx 0.6097561 \cdot d_3 + 0.2736586$. Hence, the market is arbitrage free and fixing $d_3 \in (0, 1)$ makes the market complete. The forward rate is $f_{2,3} = \ln(d_2/d_3)/(1.5 - 1) = 2\ln(d_2/d_3)$. Hence,

$$f_{2,3} = 2\ln(d_2/d_3) = 2\ln(-26 + 23.051/d_3).$$

Setting $d_3 = 0.85$ gives $d_2 = 0.951, d_4 \approx 0.792$ and $f_{2,3} \approx 0.225$.

Problem 2

Given a risk free rate $r_f \geq 0$ there is a unique tangency point $(\sigma_{tan}, \overline{r}_{tan})$ on the efficient frontier such that $(\sigma_{tan}, \overline{r}_{tan})$ is a point on a line passing through the point $(0, r_f)$. The slope of this line is the Sharpe ratio and it is maximized in this case by taking $r_f = 0$. Inspecting the figure gives the slope

$$\frac{\overline{r}_{tan} - r_f}{\sigma_{tan} - 0} = \{r_f = 0\} = \frac{\overline{r}_{tan}}{\sigma_{tan}} \approx \frac{0.085}{0.29} \approx 0.29.$$

Problem 3

The market portfolio has the vector of relative weights $(3/5, 2/5)$. Hence, the expected return and variance of the return of the market portfolio are

$$\begin{aligned} \mathrm{E}[r_M] &= \frac{3}{5}0.05 + \frac{2}{5}0.1 = 0.07, \\ \mathrm{Var}(r_M) &= \frac{3^2}{5^2}0.15^2 + \frac{2^2}{5^2}0.3^2 + 2\frac{1}{2}\frac{3}{5}\frac{2}{5}0.15 \cdot 0.3 = 0.0333. \end{aligned}$$

From the CAPM formula $\mathrm{E}[r_i] - r_f = \beta_i(\mathrm{E}[r_M] - r_f)$, $i = A, B$, we get

$$r_f = \frac{\mathrm{E}[r_i] - \beta_i \,\mathrm{E}[r_M]}{1 - \beta_i}, \quad \beta_i = \frac{\mathrm{Cov}(r_i, r_M)}{\mathrm{Var}(r_M)} \quad i = A, B.$$

Moreover,

$$\mathrm{Cov}(r_A, r_M) = \frac{3}{5}\,\mathrm{Cov}(r_A, r_A) + \frac{2}{5}\,\mathrm{Cov}(r_A, r_B) = \frac{3}{5}0.15^2 + \frac{2}{5}0.5 \cdot 0.15 \cdot 0.3 = 0.0225.$$

Hence, $\beta_A = 0.6756757$ and similarly $\beta_B = 1.486486486$. This gives $r_f = 0.008333333$, i.e. $\approx 0.83\%$.

Problem 4

(a) Minimize $\mathrm{E}[U(h_1S_1^1 + h_2S_1^2)]$ over pairs (h_1, h_2) satisfying $h_1S_0^1 + h_2S_0^2 = 100$. In this case, minimize $-\mathrm{E}[\exp\{(-h_1S_1^1 - h_2S_1^2)/100\}]$ over pairs (h_1, h_2) satisfying $h_1 + h_2 = 1$. Moreover,

$$\begin{aligned}
-\mathrm{E}\exp\left\{\frac{-h_1S_1^1 - h_2S_1^2}{100}\right\} &= -\frac{3}{10}\exp\{-1.5(h_1+h_2)\} - \frac{3}{10}\exp\{-0.5(h_1+h_2)\} \\
&\quad - \frac{2}{10}\exp\{-0.5h_1 - 1.5h_2\} - \frac{2}{10}\exp\{-1.5h_1 - 0.5h_2\} \\
&=: A + B + C + D.
\end{aligned}$$

(b) Setting $L(h_1, h_2, \lambda) = \mathrm{E}[U(h_1S_1^1 + h_2S_1^2)] + \lambda(1 - h_1 - h_2)$, computing the partial derivatives w.r.t. h_1, h_2 and setting them to zero we get

$$\begin{aligned}
&-1.5A - 0.5B - 0.5C - 1.5D - \lambda = 0, \\
&-1.5A - 0.5B - 1.5C - 0.5D - \lambda = 0.
\end{aligned}$$

This implies that $C = D$ which is equivalent to $\exp\{-0.5h_1 - 1.5h_2\} = \exp\{-1.5h_1 - 0.5h_2\}$ which is equivalent to $h_1 = h_2$. Hence, $h_1 = h_2 = 1/2$.

Problem 5

See Lecture Notes.