KTH Mathematics

Examination in SF2974 Portfolio Theory and Risk Management, January 10, 2011, 14:00–19:00.

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Allowed technical aids: calculator.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

Unless stated otherwise you may assume that it is possible to take positions corresponding to fractions of assets.

GOOD LUCK!

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Problem 1

Consider an investor who is an expected utility maximizer with a power utility function $u(x) = \sqrt{x}$. The investor has 100 Euro and has the opportunity to take long positions in a defaultable bond, a credit default swap on this bond, and in a risk-free government bond. One defaultable bond costs 96 Euro now and pays 100 Euro six months from now if the issuer does not default and 0 in case the issuer defaults. The credit default swap costs 2 Euro and pays 100 Euro six months from now if the bond issuer defaults and nothing otherwise. The risk-free bond costs 99 Euro now and pays 100 Euro six months from now. The investor believes that the default probability is 0.02. How much of the 100 Euro does the investor invest in the defaultable bond, in the risk-free bond, and in the credit default swap, respectively? (10 p)

Problem 2

Consider a company that has the option to start production of a volume $t \geq 0$ of a certain good during the next year. The company has the capital 10 to use for the production and/or to deposit on a risk-free bank account that does not pay any interest. The cost for producing a volume $t > 0$ of the good is t plus a start up cost of 5. The income from selling a volume t of the good is $5t$. The unknown demand for the good (the maximum volume the company can sell) is modeled as a random variable with distribution function $1 - x^{-2}$, $x \geq 1$.
How much should the company produce in order to maximize the expected net value at the end of next year: income from sales of the good plus money on the bank account? What is the expected net value at the end of next year when producing the optimal volume of the good? (10 p)

Problem 3

Consider the situation in Problem 2 and suppose that the company decides to produce the volume $t = 2$. Plot the distribution function of X and determine $\mathrm{VaR}_{0.1}(X)$, where X is the difference between the net value at the end of next year (income from sales of the good plus money on the bank account) and the initial available capital 10 at the beginning of the year. (10 p)

Problem 4

Consider an investor who can make a one-year investment in three risky assets and a risk-free bond that pays no interest. The investor has the initial capital $V_0 = 100'000$ Euro and wants to maximize $\mathrm{E}[V_1] - \mathrm{Var}(V_1)/V_0$, where V_1 is the value of investors portfolio at the end of the year. The vector of percentage returns on the risky assets has mean $\boldsymbol{\mu}$ and covariance matrix Σ, where

$$\boldsymbol{\mu} = \begin{pmatrix} 1.15 \\ 1.10 \\ 1.05 \end{pmatrix}, \Sigma = 0.2^2 \begin{pmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{pmatrix}, \Sigma^{-1} = \frac{1}{0.2^2} \begin{pmatrix} 1.5 & -0.5 & -0.5 \\ -0.5 & 1.5 & -0.5 \\ -0.5 & -0.5 & 1.5 \end{pmatrix}.$$

How much of the initial capital of $100'000$ Euro is invested in each of the four assets (including the risk-free bond)? Both long and short positions are allowed. (10 p)

Problem 5

A bank has sold a derivative instrument A and wants to hedge its value at time T by taking a position in an asset B and in a risk-free zero-coupon bond maturing at time T with face value 1. The expected value and standard deviation of the value of the derivative instrument A at time T is assumed to be μ_A and σ_A, respectively. The expected value and standard deviation of the value of the asset B at time T is assumed to be μ_B and σ_B, respectively. The linear correlation coefficient between the time T values of derivative A and asset B is assumed to be ρ. The hedge is constructed so that the expected value of the squared difference between the time T value of the position in the risk-free bond and asset B and the time T value of the derivative A is minimized.

Determine the positions in the risk-free bond and asset B that form the time T hedge of the derivative instrument. Both long and short positions are allowed. (10 p)

Problem 1

In the presence of the defaultable bond and the credit default swap the risk-free bond is a redundant asset since the payoff of the risk-free bond is the sum of the payoffs of the defaultable bond and the CDS. Since the price of the risk-free bond is greater than the sum of the prices of the defaultable bond and the CDS, it would be suboptimal to invest in it. The solution to

$$\begin{array}{ll} \text{maximize} & \mathrm{E}[(w_1 c_1^{-1} X_1 + w_2 c_2^{-1} X_2)^\beta] \\ \text{subject to} & w_1 + w_2 \le V_0 \\ & w_1 \ge 0, w_2 \ge 0, \end{array}$$

where $X_k \sim Be(p_k)$ and $X_1 + X_2 = 1$, is

$$w_k = V_0 \Big(\frac{c_k^\beta}{\beta p_k}\Big)^{1/(\beta-1)} \Big/ \Big\{ \Big(\frac{c_1^\beta}{\beta p_1}\Big)^{1/(\beta-1)} + \Big(\frac{c_2^\beta}{\beta p_2}\Big)^{1/(\beta-1)} \Big\}$$

Here $c_1 = 0.96$, $c_2 = 0.02$, $p_1 = 0.98$, $p_2 = 0.02$, $V_0 = 100$, $\beta = 0.5$. This gives $(w_1, w_2) \approx (98, 2)$.

Problem 2

There is an initial cost of $t + 5I\{t > 0\}$ for producing a volume $t \le 5$. This means that there is the capital $10 - t - 5I\{t > 0\}$ left to deposit on the bank account. The net value at the end of next year is therefore the income from sales plus the money on the bank account:

$$V_1 = 5\min(D, t) + (10 - t - 5I\{t > 0\}) \quad \text{for } t \in [0, 5].$$

We find that $\min(D, t) = t$ for $t \in [0, 1]$ and

$$\mathrm{E}[\min(D, t)] = \int_1^t x 2x^{-3} dx + \int_t^\infty t 2x^{-3} dx = 2 - 1/t \quad \text{for } t \in [1, 5].$$

Therefore

$$f(t) := \mathrm{E}[V_1] = \begin{cases} 10 & t = 0, \\ 4t + 5 & t \in (0, 1), \\ 5(2 - 1/t) - t + 5 & t \in [1, 5]. \end{cases}$$

In particular, $f'(t) = 4I\{t \in (0, 1)\} + (5t^{-2} - 1)I\{t \in [1, 5]\}$ for $t > 0$. Setting this expression equal to 0 and solving for t gives $t = \sqrt{5} \approx 2.236$. Since $f(t)$ is concave, $f''(t) \le 0$, for $t > 0$ we find that for $t > 0$, f has its maximum at $t = \sqrt{5}$. We find that $f(\sqrt{5}) \approx 10.764$. Since $f(0) = 10 < f(\sqrt{5})$ it is optimal to produce the volume $t = \sqrt{5}$.

Problem 3

We have $X := 5\min(D, 2) - 7$ and we find that $X \le 3$ and $\mathrm{P}(X = 3) = \mathrm{P}(D \ge 2) = 2^{-2} = 0.25$. For $x \in [-2, 3)$ we have

$$F_X(x) := \mathrm{P}(X \le x) = \mathrm{P}(D \le (x + 7)/5) = 1 - \Big(\frac{x + 7}{5}\Big)^{-2}.$$

Set $L := -X$ and note that $\mathrm{VaR}_{0.1}(X) = F_L^{-1}(0.9)$. L takes values in $[-3, 2]$, $\mathrm{P}(L = -3) = 0.25$, and for $l \in (-3, 2]$ we have

$$F_L(l) := \mathrm{P}(-X \leq l) = \mathrm{P}(X \geq -l) = \Big(\frac{7-l}{5}\Big)^{-2}.$$

Solving $F_L(l) = 0.9$ for l gives $l := F_L^{-1}(0.9) = 7 - 5 \cdot 0.9^{-1/2} \approx 1.73$. Therefore $\mathrm{VaR}_{0.1}(X) \approx 1.73$.

Problem 4

It holds that

$$V_1 = h_0 + \sum_{k=1}^{3} h_k S_1^k = w_0 R_0 + \mathbf{w}^{\mathrm{T}}\mathbf{R},$$

where w_0, w_1, w_2, w_3 are the monetary weights in the four assets, respectively. In particular, $\mathrm{E}[V_1] = w_0 R_0 + \mathbf{w}^{\mathrm{T}}\boldsymbol{\mu}$ and $\mathrm{Var}(V_1) = \mathbf{w}^{\mathrm{T}}\Sigma\mathbf{w}$. Maximizing $\mathrm{E}[V_1] - \mathrm{Var}(V_1)/V_0$ is the same as maximizing

$$\frac{V_0}{2}(w_0 R_0 + \mathbf{w}^{\mathrm{T}}\boldsymbol{\mu}) - \frac{1}{2}\mathbf{w}^{\mathrm{T}}\Sigma\mathbf{w}.$$

Here $R_0 = 1$. Thus, we want to solve

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\mathbf{w}^{\mathrm{T}}\Sigma\mathbf{w} - \frac{V_0}{2}(w_0 + \mathbf{w}^{\mathrm{T}}\boldsymbol{\mu}) \\ \text{subject to} & w_0 + \mathbf{1}^{\mathrm{T}}\mathbf{w} \leq 100'000. \end{array}$$

The solution is (see LN)

$$\begin{aligned} \mathbf{w} = \frac{V_0}{2}\Sigma^{-1}(\boldsymbol{\mu} - \mathbf{1}) &= 1.25 \cdot 10^6 \begin{pmatrix} 1.5 & -0.5 & -0.5 \\ -0.5 & 1.5 & -0.5 \\ -0.5 & -0.5 & 1.5 \end{pmatrix} \begin{pmatrix} 0.15 \\ 0.10 \\ 0.05 \end{pmatrix} \\ = 10^3 \begin{pmatrix} 187.5 \\ 62.5 \\ -62.5 \end{pmatrix}. \end{aligned}$$

The solution is $(w_0, w_1, w_2, w_3) = 10^3 \cdot (-87.5, 187.5, 62.5, -62.5)$.

Problem 5

The size of the position in asset B is

$$\frac{\sigma_A \sigma_B \rho}{\sigma_B^2} = \frac{\sigma_A}{\sigma_B}\rho.$$

The size of the position in the risk-free asset is

$$\mu_A - \frac{\sigma_A}{\sigma_B}\rho\mu_B.$$