KTH Mathematics

Examination in SF2974 Portfolio Theory and Risk Management, January 13, 2010, 8:00–13:00.

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Allowed technical aids: calculator.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

Unless stated otherwise you may assume that it is possible to buy fractions of assets.

GOOD LUCK!

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Problem 1

Consider a one year investment period and a market with n risky assets and one risk-free zero-coupon bond with a face value of \$1. The vector of relative returns of the n risky assets has finite mean and positive definite covariance matrix. Consider three portfolios A, B, and C whose relative returns have the standard deviations and expected values listed in Table 1.

Portfolio	A	B	C
Standard deviation	0.3	0.4	0.5
Expected value	0.08	0.10	0.11

Table 1: Standard deviations and expected values of three relative portfolio returns.

Suppose that at least one of the three portfolios is an efficient portfolio in the sense of the Markowitz problem (M'). What can be said about the current spot price of the one-year zero-coupon bond?

(10 p)

The following information applies to the remaining problems (2-5).

Let $\$X$ be the spot price of a certain quantity of orange juice one year from now. Consider a market with five digital options on the value X, four call options on X, and a risk-free one-year zero-coupon bond with a face value of \$1.
Suppose that X takes a value in the set $\{1, 2, 3, 4, 5\}$. The kth digital option pays \$1 if $X = k$ and nothing otherwise. The kth call option, for $k \in \{1, 2, 3, 4\}$, pays $\$(X - k)$ if $X > k$ and nothing otherwise.
Let $\$c_k$ be the current spot price of the kth digital option and let p_k be your assessment of the probability of the event $\{X = k\}$. The c_ks and p_ks are given by

$$(c_1, c_2, c_3, c_4, c_5) = (0.095, 0.285, 0.285, 0.190, 0.095),$$
$$(p_1, p_2, p_3, p_4, p_5) = (0.1, 0.15, 0.2, 0.3, 0.25).$$

The call options and the bond are priced so that there are no arbitrage opportunities.

Problem 2

You have \$1000 that you want to invest fully in a combination of long positions in the four call options and the zero-coupon bond. You want to maximize the expected value of your portfolio one year from now, according to your view of the distribution of X, with the constraint that you require the value of your portfolio one year from now to be at least \$500. How many dollars do you invest in each of the call options?

(10 p)

Problem 3

You have \$1000 that you want to invest fully in a combination of long and short positions in the fourth and fifth digital options. You want to form a portfolio that minimizes the variance of the portfolio value one year from now subject to the constraint that the expected value of the portfolio should be at least \$1400. How many dollars do you invest in each of the digital options?

(10 p)

Problem 4

You have \$1000 that you want to invest fully in a combination of long positions in the five digital options. You want to form a portfolio that maximizes the expected utility of the portfolio value one year from now. Your utility function is $u(x) = \sqrt{x}$. How many dollars do you invest in each of the digital options?

(10 p)

Problem 5

You have $V_0 = \$1000$ that you want to invest fully in a combination of long and short positions in the five digital options. You want to form a portfolio that minimizes $\mathrm{VaR}_{0.01}(V_1 - V_0/B_0)$, where V_1 is the portfolio value one year from now, and $\$B_0$ is the current spot price of the bond. How many dollars do you invest in each of the digital options?

(10 p)

Problem 1

The efficient frontier is a straight line $y = m + cx$ and the inefficient frontier is a straight line $y = m - cx$, where $m = r_f$ is the risk-free rate of return. We maximize m if one point is on the efficient frontier and another one is on the inefficient frontier and the last one is somewhere in the middle. Solving $m + 0.5c = 0.11$ and $m - 0.3c = 0.08$ gives $c = 0.0375$ and $m = 0.09125$. Other combinations give "portfolios" that are not feasible (above the efficient frontier or below the inefficient frontier). Thus we have found that $r_f \le 0.09125$. This means that $(1 - B_0)/B_0 \le 0.09125$ which gives $B_0 \ge 1/1.09125 = 0.9163803$. See Figure 1.

Problem 2

Note that $(X - k)_+ = \sum_{j=k+1}^{5} (j - k) X_j$ so the arbitrage free price π_k of $(X - k)_+$ is $\sum_{j=k+1}^{5} (j - k) c_j$. Similarly, the expected value μ_k of $(X - k)_+$ is $\sum_{j=k+1}^{5} (j - k) p_j$. Since only the bond gives money for sure, you have to buy 500 bonds. Since $1 = X_1 + \cdots + X_5$ the bond price is $c_1 + \cdots + c_5 = 0.95$. This means that $1000 - 0.95 \cdot 500 = 525$ dollars remains to invest in call options. More precisely, 525 dollars should be invested in the call option whose payoff per invested dollar has the highest expected value. Which one?

$$\frac{\mu_1}{\pi_1} = \frac{2.45}{1.805} \approx 1.36, \quad \frac{\mu_2}{\pi_2} = \frac{1.55}{0.95} \approx 1.63, \quad \frac{\mu_3}{\pi_3} = \frac{0.8}{0.38} \approx 2.11, \quad \frac{\mu_4}{\pi_4} = \frac{0.25}{0.095} \approx 2.63.$$

Thus, invest \$475 in the bond and \$525 in the call option with strike $k = 4$ (long positions).

Problem 3

Write $V_1 = h_4 X_4 + h_5 X_5$, where X_k is the payoff of the kth digital option and $h_4 c_4 + h_5 c_5 = 1000$. Moreover,

$$\begin{aligned} \operatorname{Var}(V_1) &= h_4^2 \operatorname{Var}(X_4) + h_5^2 \operatorname{Var}(X_5) + 2 h_4 h_5 \operatorname{Cov}(X_4, X_5) \\ &= h_4^2 p_4 (1 - p_4) + h_5^2 p_5 (1 - p_5) - 2 h_4 h_5 p_4 p_5. \end{aligned}$$

For now, we ignore the expected payoff constraint, solve the problem without this constraint, and finally check if the constraint is satisfied by the solution we got. The Lagrangian is

$$L(h_1, \dots, h_5, \lambda_1, \lambda_2) = \frac{1}{2} \operatorname{Var}(V_1) + \lambda (h_4 c_4 + h_5 c_5 - 1000).$$

Computing partial derivatives gives the system of equations

$$\begin{aligned} h_4 p_4 (1 - p_4) - h_5 p_4 p_5 + \lambda c_4 &= 0 \\ h_5 p_5 (1 - p_5) - h_4 p_4 p_5 + \lambda c_5 &= 0 \\ h_4 c_4 + h_5 c_5 &= 1000. \end{aligned}$$

This gives

$$\begin{aligned} & h_4 = \frac{1000}{c_4} - \frac{c_5}{c_4} h_5 \\ & h_5 \big(p_5 (1 - p_5) + 2 \frac{c_5}{c_4} p_4 p_5 + \big(\frac{c_5}{c_4} \big)^2 p_4 (1 - p_4) \big) = 1000 \big(\frac{p_4 p_5}{c_4} + \frac{p_4 (1 - p_4) c_5}{c_4^2} \big) \end{aligned}$$

which gives $(h_4, h_5) \approx (3759.4, 3007.5)$ and $(h_4c_4, h_5c_5) \approx (714.3, 285.7)$. Moreover, $\mathrm{E}[V_1] = h_4p_4 + h_5p_5 \approx 1880 > 1400$ so we have the solution to the original problem.

Problem 4

Write $V_1 = h_1X_1 + \cdots + h_5X_5$, where X_k is the payoff of the kth digital option and $h_1c_1 + \cdots + h_5c_5 = 1000$. You want to maximize

$$\mathrm{E}[u(V_1)] = \mathrm{E}[\sqrt{h_1X_1 + \cdots + h_5X_5}] = p_1\sqrt{h_1} + \cdots + p_5\sqrt{h_5}$$

subject to the budget constraint. Equivalently you want to minimize $-\mathrm{E}[u(V_1)]$ subject to the budget constraint. To solve this problem, define

$$L(h_1, \dots, h_5, \lambda) = -p_1\sqrt{h_1} - \cdots - p_5\sqrt{h_5} + \lambda(h_1c_1 + \cdots + h_5c_5 - 1000),$$

compute the partial derivatives of L with respect to the h_ks and λ, set them to zero, and solve the system of equations. The equation $\partial L/\partial h_k = 0$ has the solution $h_k = c(\lambda)(c_k/p_k)^{-2}$ and the budget constraint $\partial L/\partial \lambda = 0$ gives $c(\lambda) = 1000/(p_1^2/c_1 + \cdots + p_5^2/c_5)$. Therefore, the dollar amount invested in the kth option is

$$h_kc_k = 1000\frac{p_k^2/c_k}{p_1^2/c_1 + \cdots + p_5^2/c_5}.$$

In numbers, the dollar amounts are approximately $72.3, 54.2, 96.4, 325.3, 451.8$.

Problem 5

Write $V_1 = h_1X_1 + \cdots + h_5X_5$, where X_k is the payoff of the kth digital option and $h_1c_1 + \cdots + h_5c_5 = 1000$. If one (or more) of the h_ks are negative, then $\mathrm{VaR}_{0.01}(V_1 - V_0/B_0) > \mathrm{VaR}_{0.01}(-V_0/B_0) = 1000/B_0$, whereas $\mathrm{VaR}_{0.01}(V_1 - V_0/B_0) \le 1000/B_0$ if all the h_ks are non-negative. Therefore, the portfolio consists of only long positions. Since there are only five possible values that X can take and each event $\{X = k\}$ has a probability greater than 0.01, $\mathrm{VaR}_{0.01}(V_1 - V_0/B_0)$ is simply the least favorable outcome of $V_1 - V_0/B_0$, with a minus sign in front. The value of the least favorable outcome is maximized when $h_1 = \cdots = h_5 =: h$. Solving $hc_1 + \cdots + hc_5 = 1000$ gives $h = 1000/(c_1 + \cdots + c_5)$ and therefore the dollar amounts invested in the kth option is $hc_k = 1000c_k/(c_1 + \cdots + c_5)$. In numbers, the dollar amounts are $100, 300, 300, 200, 100$, and $\mathrm{VaR}_{0.01}(V_1 - V_0/B_0) = -\$1000(1/0.95 - 1/B_0) = \$0$.

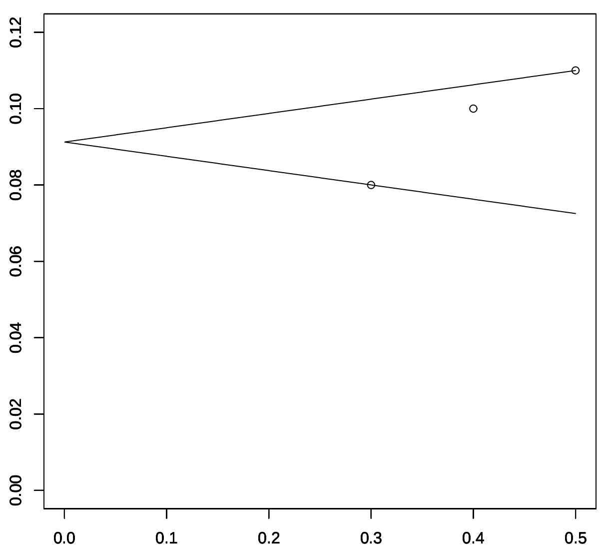

Figure 1: Possible portfolio frontier for the problem (M')