KTH Mathematics

Examination in SF2974 Portfolio Theory and Risk Management, October 20, 2009, 8:00–13:00.

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Allowed technical aids: calculator.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

There are some useful identities listed at the end.

Interest rates are continuously compounded. Unless stated otherwise you may assume that it is possible to buy fractions of assets.

Good luck!

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Problem 1

Consider the bonds specified in Table 1. Half of the annual coupon is paid every six months, until and including the time of maturity, and the first coupon payment is in six months.

	A	B	C	D
Maturity (years)	0.5	1	1.5	2
Annual coupon ($)	0	0	8	12
Bond price ($)	97.29	94.58	102.72	110.63
Face value ($)	100	100	100	100

Table 1: Specification of four different bonds.

You are facing the following payment obligation: $100000 in twenty months. Form a bond portfolio consisting of only long positions in (some of or all of) the bonds A, B, C, and D that makes you immune against small parallel shifts in the spot rate curve. The spot rates for arbitrary maturity times are determined by interpolation between spot rates for the maturity times of the four bonds.

(10 p)

Problem 2

Consider a one year investment period and a market with n risky assets. The vector $\mathbf{r}$ of relative returns of the n risky assets has mean $\boldsymbol{\mu}$ and positive definite covariance matrix Σ. Consider four portfolios with relative returns r_A, r_B, r_C, r_D with standard deviations and expected values given in Table 2.

Portfolio	A	B	C	D
Standard deviation	0.26	0.40	0.50	0.65
Expected value	0.075	0.10	0.125	0.15

Table 2: Standard deviations and expected values of four relative portfolio returns.

Determine the standard deviation and expected value of the relative return for the minimum variance portfolio under the assumption that at least three of the four portfolios A, B, C, and D are efficient portfolios in the sense of the Markowitz problem (M).

(10 p)

Problem 3

Consider two time points 0 and 1 and a risk measure ρ defined as follows. Given a position with value $\$X$ at time 1, let $\rho(X)$ be the smallest amount, in dollars, one needs to add at time 0 to the position and invest in an instrument that gives the risk-free relative return r_f at time 1, so that the new value at time 1 is for sure not smaller than a predetermined amount $\$c$.

Show that ρ is a convex measure of risk.

(10 p)

Problem 4

Consider five digital options on the value X of the two-year spot rate one year from now. Consider the partition $\mathbb{R} = A_1 \cup \cdots \cup A_5$ of the real line into disjoint intervals. The kth option pays \$1 if $X \in A_k$ and nothing otherwise. You have \$1000 that you want to invest fully in a combination of these digital options to maximize the expected value of the logarithm of the payoff in one year, according to your view of the distribution of X. The current option prices $\$c_1, \dots, \$c_5$ and your assessment of the probabilities $p_1, \dots, p_5$ of the events $\{X \in A_k\}$, $k = 1, \dots, 5$, are

$$
\begin{aligned}
(c_1, c_2, c_3, c_4, c_5) &= (0.095, 0.285, 0.285, 0.190, 0.095),\\
(p_1, p_2, p_3, p_4, p_5) &= (0.1, 0.15, 0.2, 0.3, 0.25).
\end{aligned}
$$

How many dollars do you invest in each of the options?

(10 p)

Problem 5

Consider an asset with spot price $\$S_0 = \$50$ now and random spot price $\$S_1$ at time 1. Its log-return $\log(S_1/S_0)$ is normally distributed with mean 0.05 and standard deviation 0.2. You invest \$1000 in a long position in the asset. Let $\$V_0$ and $\$V_1$ be the time 0 and time 1 values of your position. Assume that the risk-free rate of return is 0. Use the fact that $\Phi^{-1}(0.99) = 2.33$ (inverse of standard normal distribution function) to compute $\mathrm{VaR}_{0.01}(V_1 - V_0)$.

(10 p)

Useful identities

If $x \neq y$, $x \neq z$, and $y \neq z$, then

$$
\begin{pmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{pmatrix}^{-1} = \frac{1}{(y-x)(z-x)(z-y)} \begin{pmatrix} yz(z-y) & xz(x-z) & xy(y-x) \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \\ z - y & x - z & y - x \end{pmatrix}.
$$

If A is an invertible $(n \times n)$-matrix and $\mathbf{a}$ is an $(n \times 1)$-matrix, then

$$
(A + \mathbf{a}\mathbf{a}^T)^{-1} = A^{-1} - \frac{A^{-1}\mathbf{a}\mathbf{a}^T A^{-1}}{1 + \mathbf{a}^T A^{-1}\mathbf{a}}.
$$

Problem 1

The discount factors and spot rates are given by

$$\begin{pmatrix} d_{0.5} \\ d_1 \\ d_{1.5} \\ d_2 \end{pmatrix} = \begin{pmatrix} 0.9729000 \\ 0.9458000 \\ 0.9138962 \\ 0.8833436 \end{pmatrix} \text{ and } \begin{pmatrix} s_{0.5} \\ s_1 \\ s_{1.5} \\ s_2 \end{pmatrix} = \begin{pmatrix} 0.05494795 \\ 0.05572415 \\ 0.06002555 \\ 0.06202051 \end{pmatrix}.$$

The duration of the bond C is 1.443699 years and 1.844775 years for bond D. The duration of the payment obligation is $20/12 = 1.666667$ years. Linear interpolation gives the corresponding spot rate

$$s_{20/12} = s_{1.5} + \frac{s_2 - s_{1.5}}{2 - 1.5}(20/12 - 1.5) = 0.060690538$$

and the discount factor 0.9037966427. Therefore the present value of the payment obligation is \$90379.66427.
The equation

$$\begin{pmatrix} P_{\text{A}} & P_{\text{D}} \\ P_{\text{A}} D_{\text{A}} & P_{\text{D}} D_{\text{D}} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} P \\ PD \end{pmatrix}$$

has a unique solution $(a_1, a_2) = (123.0374, 708.7531)$. If instead of bonds A and D we take bonds B and D, then the solution becomes $(a_1, a_2) = (201.4720, 644.7116)$. If instead of bonds B and D we take bonds C and D, then the solution becomes $(a_1, a_2) = (390.7269, 454.1643)$. Other combinations give short positions.

Problem 2

We have $\sigma^2(\overline{r}) = c_0 + c_1\overline{r} + c_2\overline{r}^2$. We get the coefficients as follows:

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 0.075 & 0.075^2 \\ 1 & 0.125 & 0.125^2 \\ 1 & 0.15 & 0.15^2 \end{pmatrix}^{-1} \begin{pmatrix} 0.26^2 \\ 0.5^2 \\ 0.65^2 \end{pmatrix} = \begin{pmatrix} 0.2005 \\ -5.024 \\ 43.36 \end{pmatrix}$$

We find that $\sigma(0.1) = 0.3629049 < 0.4$ so portfolio B is not a frontier portfolio, but a feasible portfolio. Minimizing $\sigma^2(\overline{r})$ gives $\overline{r}_{mvp} = -c_1/(2c_2) = 0.05793358$ and $\sigma(\overline{r}_{mvp}) = 0.2344586$.

Problem 4

We want to solve the problem

$$\begin{array}{ll} \text{maximize} & \mathrm{E}[\log(m_1 c_1^{-1} I_{A_1}(X) + \cdots + m_n c_n^{-1} I_{A_n}(X))] \\ \text{subject to} & m_1 + \cdots + m_n = V_0 \\ \text{and} & m_1 \geq 0, \ldots, m_n \geq 0. \end{array}$$

This is equivalent to solving

$$\begin{array}{ll} \text{maximize} & p_1 \log(m_1) + \cdots + (1 - p_1 - \cdots - p_{n-1}) \log(V_0 - m_1 - \cdots - m_{n-1}) \\ \text{subject to} & m_1 \geq 0, \ldots, m_n \geq 0. \end{array}$$

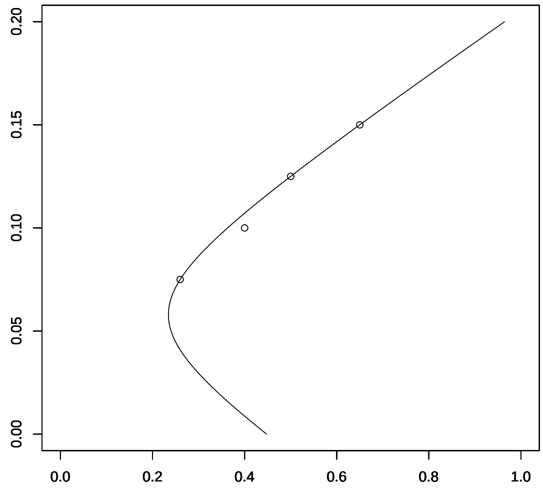

Figure 1: Portfolio frontier for the problem (M)

Let's maximize the objective function, $f = f(m_1, \dots, m_{n-1})$, and hopefully find that the inequality constraints are satisfied. The function f is concave so any local optimum is a global maximum. We find that

$$\frac{\partial}{\partial m_k} f = \frac{p_k}{m_k} - \frac{1 - p_1 - \cdots - p_{n-1}}{V_0 - m_1 - \cdots - m_{n-1}}.$$

The equation $\nabla f = \mathbf{0}$ can be written as

$$(A + \mathbf{a}\mathbf{a}^T)\mathbf{m} = \frac{V_0}{1 - p_1 - \cdots - p_{n-1}} \mathbf{1},$$

where $\mathbf{m} = (m_1, \dots, m_{n-1})^T$, $\mathbf{1} = (1, \dots, 1)^T$,

$$A = \begin{pmatrix} p_1^{-1} & & 0 \\ & \ddots & \\ 0 & & p_{n-1}^{-1} \end{pmatrix} \quad \text{and} \quad \mathbf{a} = \frac{1}{\sqrt{1 - p_1 - \cdots - p_{n-1}}} \mathbf{1}.$$

The identity

$$(A + \mathbf{a}\mathbf{a}^T)^{-1} = A^{-1} - \frac{A^{-1}\mathbf{a}\mathbf{a}^T A^{-1}}{1 + \mathbf{a}^T A^{-1} \mathbf{a}}.$$

now gives

$$\begin{aligned} \begin{pmatrix} m_1 \\ \vdots \\ m_{n-1} \end{pmatrix} &= \frac{V_0}{1 - p_1 - \cdots - p_{n-1}} \left\{ \begin{pmatrix} p_1 \\ \vdots \\ p_{n-1} \end{pmatrix} - \begin{pmatrix} p_1(p_1 + \cdots + p_{n-1}) \\ \vdots \\ p_{n-1}(p_1 + \cdots + p_{n-1}) \end{pmatrix} \right\} \\ &= V_0 \begin{pmatrix} p_1 \\ \vdots \\ p_{n-1} \end{pmatrix}. \end{aligned}$$

Problem 5

We have $S_1 = S_0 e^{\mu+\sigma Z}$, where $\mu = 0.05$, $\sigma = 0.2$, and $Z \sim \mathrm{N}(0,1)$. Moreover, $V_0 = 20S_0$ and $V_1 = 20S_1$.

$$\begin{aligned}
\mathrm{VaR}_{0.01}(V_1 - V_0) &= 20\,\mathrm{VaR}_{0.01}(S_1 - S_0) \\
&= 20(S_0 + \mathrm{VaR}_{0.01}(S_0 e^{\mu+\sigma Z}) \\
&= 1000(1 + \mathrm{VaR}_{0.01}(e^{\mu+\sigma Z}) \\
&= 1000(1 - e^{\mu+\sigma\Phi^{-1}(0.01)}) \\
&= 1000(1 - e^{\mu-\sigma\Phi^{-1}(0.99)}) \\
&\approx 340.32
\end{aligned}$$