KTH Mathematics

Examination in SF2974 Portfolio Theory and Risk Management, October 21, 2008, 14:00–19:00.

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Allowed technical aids: calculator.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

Interest rates are continuously compounded.

Good luck!

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Problem 1

A company faces a stream of payment obligations over the next three years as shown in Table 1, which also shows the current spot rates for different times to maturity. The company wants to obtain an immunization against a small parallel shift in the spot rate curve by taking long positions in two bonds. The current bond prices ($) and (Fisher-Weil) durations (years) for the three available bonds are: $(P_1, D_1) = (99, 3.7)$, $(P_2, D_2) = (99, 2.9)$ and $(P_3, D_3) = (96, 1.9)$. Determine suitable long positions in two of the three bonds. (10 p)

Time (years)	0.5	1	1.5	2	2.5	3
Payment ($1000)	100	20	80	100	150	200
Spot rates (%)	3.0	3.6	4.0	4.4	4.6	4.8

Table 1: Payment obligations in thousands of dollars and spot rates.

Problem 2

Consider an arbitrage-free market of risk-free bonds. Consider the following three bonds, each with face value 100 dollars. Bond A matures in one year from now and pays no coupons. The bond price is 95 dollars. Bonds B and C matures in four years from now and pays a coupon at the end of each year. The first coupon payment is in one year from now and the last in four years from now. Bond B pays annual coupons of 3 dollars and its current price is 90 dollars. Bond C pays annual coupons of 6 dollars and its current price is 100 dollars.

Determine the arbitrage-free forward rate for borrowing and lending dollars between the time points one year from now and four years from now. (10 p)

Problem 3

Consider a fixed time horizon and a market with three risky assets. Consider the following vectors of relative portfolio weights:

$$\mathbf{w}_1 = \begin{pmatrix} -0.3 \\ 0.6 \\ 0.7 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 0.65 \\ 0.20 \\ 0.15 \end{pmatrix}, \quad \mathbf{w}_3 = \begin{pmatrix} -1.5 \\ 2.0 \\ 0.5 \end{pmatrix}, \quad \mathbf{w}_4 = \begin{pmatrix} -1.25 \\ 1.0 \\ 1.25 \end{pmatrix}.$$

Three of the four corresponding portfolios are frontier portfolios (i.e. solutions to the Markowitz problem). We have $\mathrm{E}(\mathbf{w}_j^T\mathbf{r}) = \mu_j$ and $\mathrm{Cov}(\mathbf{w}_j^T\mathbf{r}, \mathbf{w}_k^T\mathbf{r}) = \Sigma_{jk}$, where $\mathbf{r}$ is the vector of relative asset returns and

$$\boldsymbol{\mu} = \begin{pmatrix} 0.07 \\ 0.04 \\ 0.09 \\ 0.10 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0.06570 & 0.048150 & 0.07740 & 0.083250 \\ 0.04815 & 0.070425 & 0.03330 & 0.025875 \\ 0.07740 & 0.033300 & 0.19350 & 0.121500 \\ 0.08325 & 0.025875 & 0.12150 & 0.140625 \end{pmatrix}.$$

(a) Determine the relative portfolio weights and the expected relative return of the minimum variance portfolio. (4 p)

Suppose that a risk-free asset with relative return $r_f = 0.02$ is introduced on this market.

(b) Draw the two portfolio frontiers (approximately but accurately) and determine the relative portfolio weights of the tangent portfolio by inspection of the drawn portfolio frontiers. (4 p)

(c) Determine whether it is possible for the assumptions of CAPM to hold. (2 p)

Problem 4

Consider an arbitrage-free market of n assets and let $\mathbf{r}$ be the vector of relative asset returns from today until one year from now. Consider an investor with an increasing and concave utility function U. The investor has an initial wealth of 100 Euro which is invested fully to obtain a portfolio consisting of positions in the n assets. The investor's portfolio will be held for one year. Shorting is allowed and unrestricted.

(a) Formulate, as precisely as possible, a condition for the relative portfolio weights $\mathbf{w}$ of the portfolio which is sufficient for $\mathbf{w}$ to be the vector of relative portfolio weights for the portfolio that maximizes the expected utility of the investor's wealth one year from now. (5 p)

(b) Take $U(x) = -e^{-x/100}$ and suppose that $\mathbf{r}$ is normally distributed with some mean $\boldsymbol{\mu}$ and covariance matrix Σ. What is the certainty equivalent of the investor's wealth one year from now for an arbitrary but fixed portfolio? You may want to use the fact that $\mathrm{E}(e^{\lambda Z}) = e^{\lambda^2/2}$ for a standard normally distributed Z. (5 p)

Problem 5

Consider a financial supervisor using a coherent measure of risk for determining the buffer capital requirement for the portfolios of the banks under its supervision. Bank A holds a portfolio consisting of a position in stocks and a position in interest rate securities. The supervisor requires Bank A to set aside 10 million Euro as buffer capital. Bank B holds a portfolio of stocks only, the same position as Bank A but of twice the size. It is required to hold 10 million Euro as buffer capital. Bank C holds a portfolio of interest rate securities only, the same position as Bank A but of three times the size. Its current buffer capital is 12 million Euro. Will the supervisor accept that amount? Can buffer capital be withdrawn and if not at least how much additional capital must be added? (10 p)

Problem 1

The present value and duration of the payment obligations are

$$P = 100'000 d_{0.5} + \cdots + 200'000 d_3 = 591'603.7,$$
$$D = (0.5 \cdot 100'000 d_{0.5} + \cdots + 3 \cdot 200'000 d_3)/P = 2.059668.$$

We have to find a positive solution to

$$\begin{pmatrix} P_j & P_k \\ P_j D_j & P_k D_k \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} P \\ PD \end{pmatrix},$$

where $j, k \in \{1, 2, 3\}$, $j \neq k$. Choosing $j = 2, k = 3$ gives $c_1 = 954.1432$ and $c_2 = 5178.5783$. Choosing $j = 1, k = 3$ gives $c_1 = 530.0796$ and $c_2 = 5615.8940$.

Problem 2

We need the discount factors d_k for $k = 1$ and $k = 4$ years. We have $95 = d_1 100$ which gives $d_1 = 0.95$. A long position of two type B bonds and a short position of one type C bond is equivalent to a zero coupon bond maturing in four years with face value $100 \cdot 2 - 100 = 100$ and price $90 \cdot 2 - 100 = 80$. Hence, $80 = d_4 100$ which gives $d_4 = 0.8$. The forward rate is $f_{1,4} = \ln(d_1/d_4)/(4-1) = 0.05728342$.

Problem 3

Set $\mathbf{w}(\alpha) = \alpha \mathbf{w}_2 + (1-\alpha)\mathbf{w}_4$. We find that $\mathbf{w}_1 = \mathbf{w}(0.5)$ and that there is no α satisfying $\mathbf{w}_3 = \mathbf{w}(\alpha)$. By the Two-Fund Theorem every frontier portfolio is on the form $\mathbf{w}(\alpha)$ for some α, in particular the minimum variance portfolio.

$$\mathrm{Var}(\mathbf{w}(\alpha)^T \mathbf{r}) = \alpha^2 \Sigma_{22} + (1-\alpha)^2 \Sigma_{44} + 2\alpha(1-\alpha)\Sigma_{2,4}.$$

Differentiating the right-hand side wrt α and setting the expression to zero we get

$$2\alpha\Sigma_{22} - 2(1-\alpha)\Sigma_{44} + 2(1-2\alpha)\Sigma_{2,4} = 0.$$

Solving for α we get

$$\alpha = \frac{\Sigma_{44} - \Sigma_{24}}{\Sigma_{22} + \Sigma_{44} - 2\Sigma_{24}} = 0.720339.$$

Hence, the minimum variance portfolio is $\mathbf{w}(\alpha)$ for $\alpha = 0.720339$. This gives $\mathbf{w}_{\mathrm{mvp}} = (0.1186441, 0.4237288, 0.4576271)^T$. The corresponding expected return is $0.720339\mu_2 + (1 - 0.720339)\mu_4 = 0.05677966$.
To get the tangent portfolio we inspect Figure 1 and get the expected return for the tangent portfolio. Then we solve $0.09239631 = \alpha\mu_2 + (1-\alpha)\mu_4$ which gives

$$\alpha = \frac{\mu_4 - 0.09239631}{\mu_4 - \mu_2} = 0.1267282.$$

This gives the tangent portfolio $\mathbf{w}_{\tan} = (-1.0092164, 0.8986175, 1.1105991)^T$. Since the first weight is negative we find that the market is not in equilibrium so the assumptions of CAPM cannot hold.

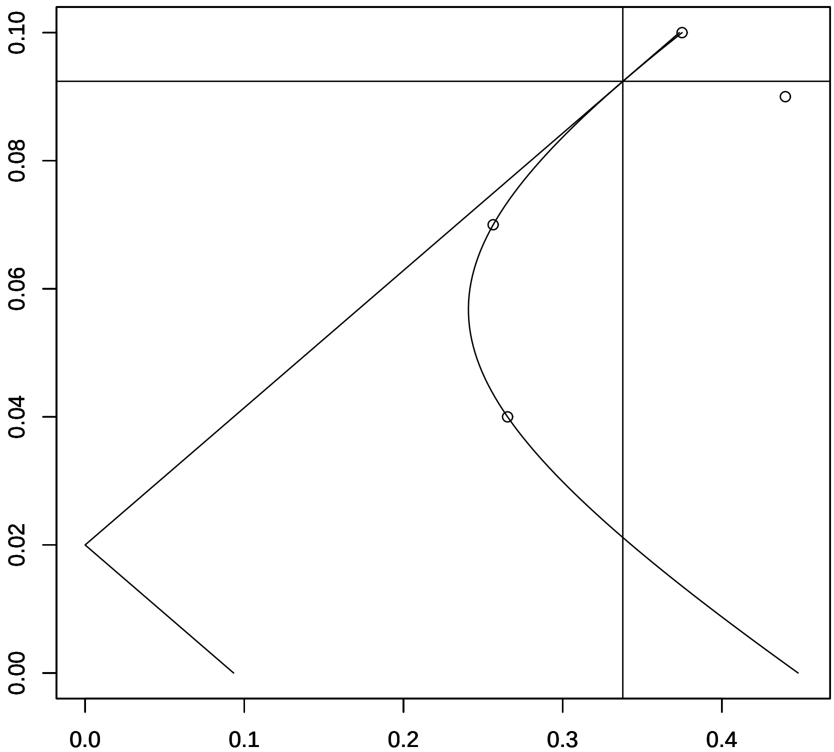

Figure 1: Portfolio frontiers, the tangent- and the given portfolios.

Problem 4

(a) With $W_0 = 100$ the condition is

$$\begin{aligned}&\mathrm{E}(U'(W_0(1+\mathbf{w}^T\mathbf{r}))W_0 r_k) = \mathrm{E}(U'(W_0(1+\mathbf{w}^T\mathbf{r}))W_0 r_1), \quad k = 2,\dots,n,\\ &\mathbf{w}^T\mathbf{1} = 1.\end{aligned}$$

(b) Let C denote the certainty equivalent satisfying

$$U(C) = \mathrm{E}(U(W_1)) = \mathrm{E}(U(W_0(1+\mathbf{w}^T\mathbf{r}))).$$

Note that $\mathbf{w}^T\mathbf{r}$ is normally distributed with mean $\mathbf{w}^T\boldsymbol{\mu}$ and variance $\mathbf{w}^T\Sigma\mathbf{w}$. Hence,

$$\begin{aligned}U(C) &= \mathrm{E}(-e^{-1-\mathbf{w}^T\boldsymbol{\mu}-\sqrt{\mathbf{w}^T\Sigma\mathbf{w}}Z})\\ &= -e^{-1-\mathbf{w}^T\boldsymbol{\mu}}\,\mathrm{E}(-e^{-\sqrt{\mathbf{w}^T\Sigma\mathbf{w}}Z})\\ &= -e^{-1-\mathbf{w}^T\boldsymbol{\mu}+\mathbf{w}^T\Sigma\mathbf{w}/2}.\end{aligned}$$

Since $U(C) = -e^{-C/100}$ we find that

$$\begin{aligned}C &= 100(1+\mathbf{w}^T\boldsymbol{\mu} - \mathbf{w}^T\Sigma\mathbf{w}/2)\\ &= \mathrm{E}(100(1+\mathbf{w}^T\mathbf{r})) - \frac{1}{200}\operatorname{var}(100(1+\mathbf{w}^T\mathbf{r}))\\ &= \mathrm{E}(W_1) - \frac{1}{200}\operatorname{var}(W_1).\end{aligned}$$

Problem 5

With $c_1 = 10$, $c_2 = 2$, $c_3 = 10$ and $c_4 = 3$ we have

$$\rho(X_1 + X_2 + c_1) = 0 \quad \text{and} \quad \rho(c_2 X_1 + c_3) = 0.$$

Hence, $\rho(X_1) = c_3/c_2$ and $c_1 = \rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$. This gives

$$\rho(c_4 X_2) = c_4\rho(X_2) \geq c_4(c_1 - c_3/c_2) = 15.$$

At least $3 = 15 - 12$ million Euro must be added.